Dynamic Quantile Models of Rational Behavior*

Luciano de Castro† Antonio F. Galvao‡

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Abstract

This paper develops a dynamic model of rational behavior under uncertainty, in which the agent maximizes the stream of the future τ-quantile utilities, for $\tau \in (0, 1)$. That is, the agent has a quantile utility preference instead of the standard expected utility. Quantile preferences have useful advantages, such as robustness and ability to capture heterogeneity. Although quantiles do not have some of the useful properties of expectations, such as linearity and the law of iterated expectations, we show that the quantile preferences are dynamically consistent. We also show that the corresponding dynamic problem yields a value function, via a fixed-point argument, and establish its concavity and differentiability. The principle of optimality also holds for this dynamic model. Additionally, we derive the corresponding Euler equation. Empirically, we show that one can employ existing generalized method of moments for estimating and testing the economic model directly from the stochastic Euler equation. Thus, the parameters of the model can be estimated using known econometric techniques and interpreted as structural objects. In addition, the methods provide microeconomic foundations for quantile regression estimation. To illustrate the developments, we construct an asset-pricing model and estimate the risk-aversion parameters across the quantiles. The results provide strong evidence of heterogeneity in the coefficients of risk aversion and discount factor.

Keywords: Quantile utility, dynamic programing, generalized method of moments, quantile regression, asset pricing.

JEL: C22, C61, E20, G12

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†Department of Economics, University of Iowa, Iowa City, USA. decastro.luciano@gmail.com

‡Department of Economics, University of Iowa, Iowa City, USA. antonio-galvao@uiowa.edu

All models are wrong. Some are useful.

George Box

1 Introduction

Recently, there has been a great effort to incorporate heterogeneity into dynamic economic models and econometrics.[1] We contribute to this effort by developing a new dynamic model for an individual, who, when selecting among uncertain alternatives, chooses the one with the highest τ-quantile of the utility distribution for $\tau \in (0, 1]$, instead of the standard expected utility. This quantile preference model is tractable, simple to interpret, and substantially broadens the scope of economic applications, because it is robust to fat tails and allows to account for heterogeneity through the quantiles.

Quantile preferences were first studied by Manski (1988) and axiomatized by Chambers (2009) and Rostek (2010). Manski (1988) develops the decision-theoretic attributes of quantile maximization and examines risk preferences of quantile maximizers. In the context of preferences over distributions, Chambers (2009) shows that monotonicity, ordinal covariance, and continuity characterize quantile preferences. Rostek (2010) axiomatizes the quantile preference in Savage (1954)'s framework, using a 'typical' consequence (scenario). In addition, Rostek (2010) discusses the advantages of the quantile preferences, such as robustness and heterogeneity. Thus, quantile preferences are a useful alternative to the expected utility, and a plausible complement to the study of rational behavior under uncertainty.[2]

This paper initiates the use of quantile preferences in a dynamic economic setting by providing a comprehensive analysis of a dynamic rational quantile model. As a first step in the development, we introduce dynamic programming for intertemporal decisions whereby the economic agent maximizes the present discounted value of the stream of future τ-quantile utilities by choosing a decision variable in an feasible set. Our first main result establishes dynamic consistency of the quantile preferences, in the sense commonly adopted in decision theory. Second, we show that the optimization problem leads to a contraction, which therefore has a unique fixed-point. This fixed point is the value function of the problem and satisfies the Bellman equation. Third, we prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Fourth, we show that the principle of optimality holds. Fifth, using these results, we derive the corresponding Euler equation for the infinite horizon problem.

We note that the theoretical developments and derivations in this paper are of independent interest. From a theoretical point of view, the main results for the dynamic quantile model

[1] See, among others, Krusell and Smith (2006), Heathcote, Storesletten, and Violante (2009), and Guvenen (2011) for economic models and Matzkin (2007) and Browning and Carro (2007) for econometrics.

[2] Quantile preferences can be associated with Choquet expected utility (see, e.g., Chambers (2007); Bassett, Koenker, and Kordas (2004)). The method of Value-at-Risk, which is widespread in finance, also is an instance of quantiles (see, e.g., Engle and Manganelli (2004)).

– dynamic consistency, value function, principle of optimality, and Euler equation – are parallel to those of the expected utility model. However, because quantiles do not share all of the convenient properties of expectations, such as linearity and the law of iterated expectations, the generalizations of the results from expected utility to quantile preference are not straightforward.

The derivation of the Euler equation is an important feature of this paper because it allows to connect economic theory with empirical applications. We show that the Euler equation has a conditional quantile representation and relates to quantile regression econometric methods. The Euler equation, which must be satisfied in equilibrium, implies a set of population orthogonality conditions that depend, in a nonlinear way, on variables observed by an econometrician and on unknown parameters characterizing the preferences. Thus, empirically, one can employ existing general econometric methods such as (non-smooth) generalized method of moments (GMM) for estimating and testing the parameters of the model. In this fashion, these parameters can be interpreted as structural objects, and practical inference is simple to implement. In addition, varying the quantiles τ enables one to empirically estimate a set of parameters of interest as a function of the quantiles. This approach allows learning about the potential underlying parameter heterogeneity among the different τ-quantiles. We note that the theoretical methods do not impose restrictions across quantiles, and thus the parameter estimates might (or might not) reveal the underlying heterogeneity. Therefore, our methods provide microeconomic foundations for quantile regression, and could be interpreted as providing a test for the empirical relevance of heterogeneity.

Finally, we illustrate the methods with an asset-pricing model, which is central to contemporary economics and finance, and has been extensively used.[3] We use a variation of Lucas (1978)'s model where the economic agent decides on how much to consume and save by maximizing a quantile utility function subject to a linear budget constraint. We solve the dynamic problem and obtain the Euler equation. Following a large body of literature, we specify a constant relative risk aversion utility function and estimate the implied risk aversion and discount factor parameters. The empirical results document strong evidence of heterogeneity in both the coefficient of risk aversion and discount factor across quantiles. An interesting result is that the coefficient of risk aversion is relatively larger for the lower quantiles and smaller for the upper quantiles. This outcome is as predicted by the notion of risk studied by Manski (1988). These results help to shed light on the equity premium puzzle and make it possible to reconcile the relative large spread observed between the risk-free and risky assets with the large relative risk aversion of an agent maximizing lower quantiles.

More broadly, this paper contributes to the literature by robustifying economic and policy design, and capturing potential heterogeneity by varying the quantiles τ. The proposed methods could be applied to any dynamic economic problem, substituting the standard max-

[3]See, among others, Hansen and Singleton (1982), Mehra and Prescott (1985), Cochrane (2005), Mehra (2008), Mehra and Prescott (2008) and Ljungqvist and Sargent (2012), and references therein.

imization of expectation by maximization of the quantile objective function. Since dynamic economic models are now routinely used in many fields, such as macroeconomics, finance, international economics, public economics, industrial organization and labor economics, among others, the proposed methods expand the scope of economic analysis and empirical applications, providing an interesting alternative to the expected utility models.

The remaining of the paper is organized as follows. Section 2 presents definitions and basic properties of quantiles. Section 3 describes the dynamic economic model and presents the main theoretical results. Section 4 discusses the estimation and inference. Section 5 illustrates the empirical usefulness of the the new approach by applying it to the asset pricing model. Finally, Section 6 concludes. We relegate all proofs to the Appendix.

1.1 Review of the Literature

This paper has a broad scope and relates to a number of streams of literature in economic theory and econometrics.

First, the paper relates to the extensive literature on dynamic nonlinear rational expectations models. Many models of dynamic maximization that use expected utility have been proposed and discussed. These models have been workhorses in several economic fields. We refer the reader to more comprehensive works, such as Stokey, Lucas, and Prescott (1989) and Ljungqvist and Sargent (2012). We extend this literature by replacing expected utility with quantile utility. Another related segment of the literature studies recursive utilities. We refer the reader to Epstein and Zin (1989), Marinacci and Montrucchio (2010), and Remark 3.11 below for further discussions.

Second, this paper is related to the rich literature on economic models with heterogeneity. Heckman (2001), Blundell and Stoker (2005), Krusell and Smith (2006), and Guvenen (2011) provide reviews of the main ideas on heterogeneity and aggregation. Dynamic models with heterogeneity typically feature individual-specific uncertainty that stems from fluctuations in labor earnings, health status, and portfolio returns, among others. Virtually all of these models rely on the expected utility framework and capture heterogeneity in a variety of ways. Part of the literature allows for heterogeneity of the economic variables and shocks, but restricts the parameters of interest – parameters that characterize the preference, for example – to be homogeneous (see, e.g., Krusell and Smith (1998), Dynan (2000), Heaton and Lucas (2008)). Another body of the literature encompasses heterogeneity by allowing the parameters to vary in a small set – as a binary set, for example – (see, e.g., Mazzocco (2008), and Guvenen (2009)). Yet another stream of the literature incorporates more general heterogeneity in the parameters of interest, but imposes ad-hoc parametric restrictions on them – see, e.g., Herranz, Krasa, and Villamil (2015). In this paper, we contribute to this literature by using the quantile preference instead of the expected utility, which allows to account for heterogeneity through the quantiles.

Third, the paper relates to an extensive literature on estimating Euler equations. Since the contributions of Hall (1978), Lucas (1978), Hansen and Singleton (1982), and Dunn and Singleton (1986) it has become standard in economics to estimate Euler equations based on conditional expectation models. There are large bodies of literature in micro and macroeconomics on this subject. We refer the reader to Attanasio and Low (2004) and Hall (2005), and the references therein, for a brief overview. The methods in this paper derive a Euler equation that has a conditional quantile function representation and estimate it using existing econometric methods.

Finally, this paper is related to the quantile regression (QR) literature, for which there is a large body of work in econometrics.[4] Koenker and Bassett (1978) developed QR methods for estimation of conditional quantile functions. These models have provided a valuable tool in economics and statistics to capture heterogeneous effects, and for robust inference when the presence of outliers is an issue (see, e.g., Koenker (2005)). QR has been largely used in program evaluation studies (Chernozhukov and Hansen (2005) and Firpo (2007)). Quantiles are employed for identification of nonseparable models (Chesher (2003) and Imbens and Newey (2009)), nonparametric identification and estimation of nonadditive random functions (Matzkin (2003)), and testing models with multiple equilibria (Echenique and Komunjer (2009)). This paper contributes to the effort of providing microeconomic foundations for QR by developing a dynamic optimization decision model that generates a conditional quantile restriction (Euler equation).

2 Preliminaries

This section introduces basic concepts considered in the paper. Subsection 2.1 defines quantiles and establishes well-known basic results that are useful later. Subsection 2.2 introduces the one-period quantile preferences that will substitute the standard expected utility preferences in our analysis. Subsection 2.3 briefly defines the notion of risk associated with the quantile preferences.

2.1 Quantiles

Let X be a random variable, with c.d.f. $F(\alpha) \equiv \Pr[X \leqslant \alpha]$. The quantile function $Q : [0,1] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the generalized inverse of F, that is,

$$Q(\tau) \equiv \begin{cases} \inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant \tau\}, & \text{if } \tau \in (0,1] \\ \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}, & \text{if } \tau = 0. \end{cases} \tag{1}$$

[4] This paper is also related to an econometrics literature on identification, estimation, and inference of general (non-smooth) conditional moment restriction models. We refer the reader to, among others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), Chen, Chernozhukov, Lee, and Newey (2014), and Chen and Liao (2015).

The definition is special for $\tau = 0$ so that the quantile assumes a value in the support of Y.[5] It is clear that if F is invertible (for instance, if F is continuous and strictly increasing), its generalized inverse coincide with the inverse, that is, $Q(\tau) = F^{-1}(\tau)$. Usually, it will be important to highlight the random variable to which the quantile refers. In this case we will denote $Q(\tau)$ by $Q_\tau[X]$. For convenience, throughout the paper we will focus on $\tau \in (0, 1)$, unless explicitly stated.

In Lemma 7.1 in the appendix, we develop some useful properties of quantiles, such as the fact that it is left-continuous and $F(Q(\tau)) \geqslant \tau$. Another well-known and useful property of quantiles is "invariance" with respect to monotonic transformations, that is, if $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, then

$$Q_\tau[g(X)] = g\left(Q_\tau[X]\right). \tag{2}$$

For $\tau \in (0, 1]$, the conditional quantile of W with respect to Z is defined as:

$$Q_\tau[w|z] \equiv \inf\{\alpha \in \mathbb{R} : \Pr\left([W \leqslant \alpha]\,|Z = z\right) \geqslant \tau\}. \tag{3}$$

Lemma 7.2, in the appendix, generalizes (2) to conditional quantiles. More precisely, Lemma 7.2 proves that if $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ is non-decreasing and left-continuous in $Z \in \mathcal{Z}$, then,

$$Q_\tau[g(\theta, \cdot)|Z = z] = g\left(\theta, Q_\tau[W|Z = z]\right). \tag{4}$$

This property is repeatedly used in the rest of the paper.

2.2 Quantile Preference

Expected utility is the widely used preference in economics and econometrics. To contextualize the difference between the expected utility and the quantile preferences, let $\mathcal{R}$ denote the set of random variables (lotteries). We say that the functional $\varphi : \mathcal{R} \to \mathbb{R}$ *represents the preference* $\succeq$ if for all $X, Y \in \mathcal{R}$ we have:

$$X \succeq Y \iff \varphi(X) \geqslant \varphi(Y). \tag{5}$$

In von-Neumann-Morgenstern's expected utility, $\varphi(X) = E[u(X)]$. To be more specific, von-Neumann-Morgenstern theorem states that $\succeq$ satisfies completeness, transitivity, continuity and independence if and only if there exists an utility function u such that[6]

$$X \succeq Y \iff E[u(X)] \geqslant E[u(Y)]. \tag{6}$$

This paper departs from this standard preference by adopting the quantile preferences,

[5] Indeed, $\inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant 0\} = -\infty$, no matter what is the distribution.
[6] See Kreps (1988) for more details.

where the functional φ in (5) is given by a quantile function, that is, $\varphi(X) = Q_\tau[u(X)]$, so that:

$$X \succeq Y \iff Q_\tau[u(X)] \geqslant Q_\tau[u(Y)]\,. \tag{7}$$

Manski (1988) was the first to study this preference, which was recently axiomatized by Chambers (2009) and Rostek (2010). Rostek (2010) axiomatized the quantile preferences in the context of Savage (1954)'s subjective framework. Rostek (2010) modifies Savage's axioms to show that they are equivalent to the existence of a $\tau \in (0,1)$, probability measure and utility function such that the functional φ in equation (5) is a quantile function.[7] In contrast, the utility function and the probability distributions are in some sense already fixed in Chambers (2009)'s approach. He shows that the preference satisfies monotonicity, ordinal covariance, and continuity if and only if (7) holds, that is, the preference is a quantile preference; see his paper for more details.[8]

2.3 Risk in the Quantile Model

Another interesting property of the quantile preference is the relationship of the risk attitude with respect to the τ, identified by Manski (1988). The following result by Manski (1988) establishes the connection between the risk attitude and quantiles; see also Rostek (2010, section 6.1) for discussion, definitions and details.

Theorem 2.1 (Manski, 1988)**.** *$\succeq^{\tau'}$ is more risk-averse than $\succeq^{\tau}$ if and only if $\tau' < \tau$.*

Thus, a decision maker that maximizes a lower quantile is more risk-averse than one who maximizes a higher quantile. In other words, the "risk-aversion" (in this definition) decreases with the quantile. In Section 5 below, we empirically obtained this result in the context of asset pricing.

3 Economic Model and Theoretical Results

This section describes a dynamic economic model and develops a dynamic program theory for quantile preferences. We try to follow closely the developments of Stokey, Lucas, and Prescott (1989, chapter 9). We begin in subsection 3.1 by extending the quantile preference to a dynamic environment, suitable for our analysis. Subsection 3.2 states and discusses the assumptions used for establishing the main results. Subsection 3.3 establishes the existence of recursive functions, necessary to complete the definition of the preferences. Subsection 3.4 shows that the preference is dynamically consistent. In subsection 3.5 we establish the existence of the

[7] If $\tau \in \{0,1\}$, the statement is more complex; see her paper for details.

[8] Since the upper semicontinuity property is a technical condition and first-order stochastic dominance is a very weak and reasonable property, also satisfied by expected utility, the really important property is invariance with respect to monotonic transformations. We have stated this property before in equation (2). Thus, this property could be considered the essence of the quantile preference considered here.

value function and its differentiability. Subsection 3.6 states and proves, in our context, the Bellman's Principle of Optimality, which allows to pass from plans to single period decisions and vice-versa, thus establishing that the value function corresponds to the original dynamic problem in a precise sense. Subsection 3.7 derives the Euler equation associated to this dynamic problem, which describes the agents behavior and is useful for the econometric part of the paper. Finally, subsection 3.8 illustrates the theory with an example of the one-sector growth model.

The main results in this section are generalizations to the quantile preferences' case of the corresponding ones in Stokey, Lucas, and Prescott (1989), which focus on expected utility. First, they increase the scope of potential applications of economic models substantially by using quantile utility. Second, the generalizations are of independent interest. The demonstrations are not routine since quantiles do not possess several of the convenient properties of expectations, such as linearity and the law of iterated expectations.

3.1 Dynamic Environment and Dynamic Quantile Preference

Section 2.2 introduced and discussed the quantile preferences with respect to single period uncertainty. We adopt this preference in a dynamic environment. In such an environment, the random variable whose quantile the decision-maker/consumer is interested is given by a stream of future consumption. To describe this more formally, we introduce now a dynamic setting that will be used in the rest of the paper.

3.1.1 States and Shocks

Let $\mathcal{X} \subset \mathbb{R}^p$ denote the state space, and $\mathcal{Z} \subseteq \mathbb{R}^k$ the range of the shocks (random variables) in the model. Let $x_t \in \mathcal{X}$ and $z_t \in \mathcal{Z}$ denote, respectively, the state and the shock in period t, both of which are known by the decision maker at the beginning of period t. We may omit the time indexes for simplicity, when it is convenient. Let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ (t-times, for $t \in \mathbb{N}$), $\mathcal{Z}^\infty = \mathcal{Z} \times \mathcal{Z} \times \cdots$ and $\mathbb{N}^0 \equiv \mathbb{N} \cup \{0\}$. Given $z \in \mathcal{Z}^\infty$, $z = (z_1, z_2, ...)$, we denote $(z_t, z_{t+1}, ...)$ by ${}_t z$ and $(z_t, z_{t+1}, ..., z_{t'})$ by ${}_t z_{t'}$. A similar notation can be used for $x \in \mathcal{X}^\infty$.

The random shocks will follow an time-invariant (stationary) Markov process. More precisely, a probability density function (p.d.f.) $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ establishes the dependence between Z_t and Z_{t+1}, such that the process is invariant with respect to t. For simplicity of notation, we will frequently represent Z_t and Z_{t+1} by Z and W, respectively. We will assume that f and $\mathcal{Z}$ satisfy standard assumptions, as explicitly stated below in section 3.2.

For any topological space W, we will denote by $\sigma(W)$ the σ-algebra generated by its open sets. For each $z \in \mathcal{Z}$ and $A \in \sigma(\mathcal{Z})$, define

$$K(z, A) \equiv \int_A f(w|z) dw, \tag{8}$$

where $f(w|z) = \frac{f(z,w)}{\int_{\mathcal{Z}} f(z,w)dw}$. Thus, K is a probabilistic kernel, that is, (i) $z \mapsto K(z, A)$ is measurable for every $A \in \sigma(\mathcal{Z})$; and (ii) $A \mapsto K(z, A)$ is probability measure for every $z \in \mathcal{Z}$. In other words, K represents a conditional probability, and we may emphasize this fact by writing $K(A|z)$ instead of $K(z, A)$. We will also abuse notation by denoting $K(z, \{z' : z' \leqslant w\})$ simply by $K(w|z)$.

3.1.2 Plans and Preferences

At the beginning of period t , the decision maker knows the current state x_t and learns the shock z_t and decides (according to preferences defined below) the future state $x_{t+1} \in \Gamma(x_t, z_t) \subset \mathcal{X}$, where $\Gamma(x, z)$ is the constraint set.[9] From this, we can define plans as follows:

Definition 3.1. *A plan π is a profile $\pi = (\pi_t)_{t\in\mathbb{N}}$, where for each $t \in \mathbb{N}$, π_t is a measurable function from $\mathcal{X} \times \mathcal{Z}^t$ to $\mathcal{X}$.*[10]

The interpretation of the above definition is that a plan $\pi_t(x_t, z^t)$ represents the choice that the individual makes at time t, upon observing the current state x_t and the sequence of previous shocks z^t. The following notation will simplify statements below.

Definition 3.2. *Given a plan $\pi = (\pi_t)_{t\in\mathbb{N}} \in \Pi$, $x \in \mathcal{X}$ and realization $z^\infty = (z_1, ...) \in Z^\infty$, the sequence associated to (x, z^∞) is the sequence $(x_t^\pi)_{t\in\mathbb{N}^0} \in \mathcal{X}^\infty$ defined recursively by $x_1^\pi = x$ and $x_t^\pi = \pi_{t-1}(x_{t-1}^\pi, z^{t-1})$, for $t \geqslant 2$. Similarly, given $\pi \in \Pi$, $(x, z^t) \in \mathcal{X} \times Z^t$, the t-sequence associated to (x, z^t) is $(x_l^\pi)_{l=1}^t \in \mathcal{X}^t$ defined recursively as above.*

We may write $x_t^\pi(\cdot)$, $x_t^\pi\left(x, z^t\right)$ or $x_t^\pi\left(x, z^\infty\right)$ to emphasize that x_t^π depends on the initial state x and on the sequence of shocks z^∞, up to time t.

Definition 3.3. *A plan π is feasible from $(x, z) \in \mathcal{X}$ if $\pi_t(x_t^\pi, z^t) \in \Gamma\left(x_t^\pi, z_t\right)$ for every $t \in \mathbb{N}$ and $z^\infty \in \mathcal{Z}^\infty$, such that $x_1^\pi = x$ and $z_1 = z$.*

We denote by $\Pi(x, z)$ the set of feasible plans from $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Let Π denote the set of all feasible plans from some point, that is, $\Pi \equiv \cup_{(x,z)\in\mathcal{X}\times\mathcal{Z}}\Pi(x, z)$.

The agent's preference in period t is represented by a function $V_t : \Pi \times \mathcal{X} \times Z^t \to \mathbb{R}$, that will be specified below. Let Ω_t represent all the information revealed up to time t.[11] We assume that in time t with revealed information Ω_t, the consumer/decision-maker has a preference $\succeq_{t,\Omega_t}$ over plans $\pi, \pi' \in \Pi(x, z)$ defined as follows:

$$\pi' \succeq_{t,x,\Omega_t} \pi \iff V_t(\pi', x, z^t) \geqslant V_t(\pi, x, z^t). \tag{9}$$

[9]This model is very close to the one discussed in Stokey, Lucas, and Prescott (1989, Chapter 9). There are different, slightly more complicated dynamic models where the state is not chosen by the decision maker, but defined by the shock. The arguments in the current model can be extended to those models when preferences are expected utility, as Stokey, Lucas, and Prescott (1989, Chapter 9) discuss. In our setup, this extension may be more involved.

[10]In the expressions below, $\pi_0(z^0)$ should be understood as just $\pi_0 \in \mathcal{X}$.

[11]With the knowledge of a fixed π, Ω_t reduces to the initial state x_1 and the sequence of shocks z^t. More generally, we could take the sequence of states and shocks (x^t, z^t).

A special case of our specification so far is, of course, the standard expected utility:

$$V_t(\pi, x, z^t) = \mathrm{E}\left[\sum_{s \geqslant t} \beta^{s-t} u(x_s^\pi(x, Z^s), x_{s+1}^\pi(x, Z^s), Z_s) \middle| Z^t = z^t\right], \tag{10}$$

where $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is the current-period utility function. That is, $u(x, y, z)$ denotes the instantaneous utility obtained in the current period when $x \in \mathcal{X}$ denotes the current state, $y \in \mathcal{X}$, the choice in the current state, and $z \in \mathcal{Z}$, the current shock. Note that the Markov assumption allows to substitute the expected conditional on Z^t above by an expectation conditional only on Z_t.

It is important to realize that the functions V_t defined by (10) satisfy the following recursive equation:

$$V_t(\pi, x, z^t) = u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta \mathrm{E}\left[V_{t+1}(\pi, x, (Z^t, Z_{t+1}))\middle| Z^t = z^t\right]. \tag{11}$$

Koopmans (1960), Lucas and Stokey (1984), Epstein and Zin (1989) and more recently Marinacci and Montrucchio (2010) worked with a generalization of this recursive equation.[12] More specifically, Marinacci and Montrucchio (2010) define an aggregator function W and a certainty equivalent function C that allows us to generalize (11) to:

$$V_t(\cdot) = W\left(u(\cdot), \mathrm{C}\left[V_{t+1}(\cdot)\right]\right). \tag{12}$$

Our preferences are based on (12), where we use the same W as in the standard case, that is, $W(a, b) = a + \beta b$, and just substitute the certainty equivalent function C, which is the expectation $\mathrm{E}[\cdot]$ in (11), by the quantile function $Q_\tau[\cdot]$. That is, we impose:

$$V_t(\pi, x, z^t) = u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta Q_\tau\left[V_{t+1}(\pi, x, (Z^t, z_{t+1}))\middle| Z^t = z^t\right]. \tag{13}$$

In section 3.3 below, we explicitly define a sequence of functions V_t that satisfy (13) and will specify the preferences (9).

3.2 Assumptions

Now we state the assumptions used for establishing the main results. We organize the assumptions in two groups. The first group collects basic assumptions, which will be assumed throughout the paper, even if they are not explicitly stated. The second group of assumptions will be used only to obtain special, desirable properties of the value function.

[12] Koopmans (1960) and Lucas and Stokey (1984) are restricted to the case without uncertainty. Epstein and Zin (1989) and Marinacci and Montrucchio (2010) deals with the case of uncertainty and issues related to Kreps and Porteus (1978), which are not central to our investigation. Remark 3.11 discusses these further developments and the relationship to this paper.

Assumption 1 (Basic)**.** *The following is maintained throughout the paper:*

(i) $\mathcal{Z} \subseteq \mathbb{R}^k$ *is convex;*

(ii) $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ *is continuous, symmetric and* $f(z, w) > 0$*, for all* $(z, w) \in \mathcal{Z} \times \mathcal{Z}$*;*[13]

(iii) $\mathcal{X} \subset \mathbb{R}^p$ *is convex;*

(iv) $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is continuous and bounded;*

(v) The correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ *is continuous, with nonempty, compact, convex values.*

Note that Assumption 1(i) allows an unbounded multidimensional Markov process, requiring only that the support is convex. Assumption 1(ii) imposes continuity of f, the pdf that establishes the dependence between Z_t and Z_{t+1} and requires it to be strictly positive in the support of the Markov process, $\mathcal{Z}$. The state space $\mathcal{X}$ is not required to be compact, but only convex by Assumption 1(iii). Assumption 1(iv) is the standard continuity assumption. Assumption 1(v) and the continuity of u required in Assumption 1(iv) guarantee that an optimal choice always exist.

For some results we will also require differentiability, concavity and monotonicity assumptions.

Assumption 2 (Differentiability, Concavity and Monotonicity)**.** *The following holds:*

(i) $\mathcal{Z} \subseteq \mathbb{R}$ *is an interval;*

(ii) If $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leqslant z'$*, then:*

$$\int_{\mathcal{Z}} h(\alpha) f(\alpha|z) d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha) f(\alpha|z') d\alpha; \tag{14}$$

(iii) $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is* C^1*, strictly concave in the first two variables and strictly increasing in the last variable;*[14]

(iv) For every $x \in \mathcal{X}$ *and* $z \leqslant z'$*,* $\Gamma(x, z) \subseteq \Gamma(x, z')$*;*

(v) For all $z \in \mathcal{Z}$ *and all* $x, x' \in \mathcal{X}$*,* $y \in \Gamma(x, z)$ *and* $y' \in \Gamma(x', z)$ *imply*

$$\theta y + (1 - \theta) y' \in \Gamma[\theta x + (1 - \theta) x', z], \text{ for all } \theta \in [0, 1].$$

[13] Symmetry guarantees stationarity, since $\Pr([Z_1 \in A]) = \int_{\mathcal{Z}} \int_A f(z_1, z_2) dz_1 dz_2 = \int_A \int_{\mathcal{Z}} f(z_1, z_2) dz_1 dz_2 = \Pr([Z_2 \in A])$.

[14] Strict concavity in the first two variables means that for all $z \in \mathcal{Z}$, $\alpha \in (0, 1)$ and $(x_0, y_0) \neq (x_1, y_1)$, we have $u(x_\alpha, y_\alpha, z) > \alpha u(x_0, y_0, z) + (1 - \alpha) u(x_1, y_1, z)$, where $x_\alpha = \alpha x_0 + (1 - \alpha) x_1$ and y_α is similarly defined.

To work with monotonicity, we restrict the dimension of the Markov process to $k = 1$ in Assumption 2(i). Assumptions 2(ii) $-$ 2(v) are standard conditions on dynamic models (see, e.g., Assumptions 9.8 $-$ 9.15 in Stokey, Lucas, and Prescott (1989)). Assumption 2(ii) implies that whenever $z \leqslant z'$,

$$K(w|z') = \int_{\{\alpha \in \mathcal{Z}:\alpha \leqslant w\}} f(\alpha|z')d\alpha \leqslant \int_{\{\alpha \in \mathcal{Z}:\alpha \leqslant w\}} f(\alpha|z)d\alpha = K(w|z), \tag{15}$$

for all w.[15] In other words, $K(\cdot|z')$ first-order stochastically dominates $K(\cdot|z)$. Assumption 2(iii) allows us to establish the continuity and differentiability of the value function. Assumption 2(iv) only requires the monotonicity of the choice set. Assumption 2(v) implies that $\Gamma(s)$ is a convex set for each $s \in \mathcal{S}$, and that there are no increasing returns.

It should be noted that monotonicity also is important for econometric reasons. Indeed, Matzkin (2003, Lemma 1, p. 1345) shows that two econometric models are observationally equivalent if and only if there are strictly increasing functions mapping one to another. Thus, in a sense, the quantile implied by a model is the essence of what can be identified by an econometrician.

3.3 The Sequence of Recursive Functions

In this section, we define the sequence of functions V_t that satisfy (13) and specify the preferences (9). For this, we need to define a transformation. Let $\mathcal{C}$ denote the set of bounded continuous functions from $\mathcal{X} \times \mathcal{Z}$ to $\mathbb{R}$, endowed with the sup norm. It is well known that $\mathcal{C}$ is a Banach space. Let us fix $\pi \in \Pi$ and $\tau \in (0, 1]$, and define the transformation $\mathbb{T}^\pi : \mathcal{C} \to \mathcal{C}$ (the dependency on τ is omitted) by the following:

$$\mathbb{T}^\pi(V)(x, z) = u\,(x_1^\pi, x_2^\pi, z_1) + \beta Q_\tau[V(x_2^\pi, Z_2))|Z_1 = z], \tag{16}$$

where $(x_1^\pi, z_1) = (x, z)$ and $x_2^\pi = \pi(x, z)$. We show that the image of $\mathbb{T}^\pi$ is indeed in $\mathcal{C}$ continuous and that $\mathbb{T}^\pi$ is a contraction and, therefore, has a unique fixed point:

Theorem 3.4. *$\mathbb{T}^\pi(V)$ is continuous on $\mathcal{X} \times \mathcal{Z}$. $\mathbb{T}^\pi$ is a contraction and has a unique fixed point, denoted $V^\pi : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$.*

Now we can define V_t as follows:

$$V_t(\pi, x, z^t) = V^\pi(x_t^\pi, z_t),$$

where $(x_l^\pi)_{l=1}^t$ is the associated t-sequence to (x, z^t) (see definition 3.2). From the fact that V^π is the unique fixed point of $\mathbb{T}^\pi$, it is clear that (13) holds. This completes the definition of the preferences (9).

[15] To obtain (15), it is enough to use $h(z) = -1_{\{\alpha \in \mathcal{Z}:\alpha \leqslant w\}}(z)$ in (42).

It is possible to write V^π in a more explicit form. For this, let us define

$$
\begin{aligned}
V^n(x,z) &= u(x_1^\pi, x_2^\pi, z_1) + Q_\tau\Big[\beta u(x_2^\pi, x_3^\pi, z_2) + Q_\tau\Big[\beta^2 u(x_3^\pi, x_4^\pi, z_3) + \ldots \\
&\qquad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^\pi, x_{n+2}^\pi, z_n)\Big| Z_n = z_n\Big] \ldots \Big| Z_1 = z\Big] \\
&= \sum_{t=0}^{n} Q_\tau^t\Big[\beta^t u(x_{t+1}^\pi, x_{t+2}^\pi, z_t)\Big| Z_t = z_t\Big], \qquad (17)
\end{aligned}
$$

where the expression in the last line is just a short notation (actually, an abuse of notation) for the previous lines. With this definition, we obtain:

Proposition 3.5. $V^\pi(x,z) = \lim_{n\to\infty} V^n(x,z)$.

Thus, we can use the following (abuse of) notation:

$$
\begin{aligned}
V^\pi(x,z) &= \sum_{t=0}^{\infty} Q_\tau^t\Big[\beta^t u(x_{t+1}^\pi, x_{t+2}^\pi, z_t)\Big| Z_t = z_t\Big] \qquad (18) \\
&= u(x_1^\pi, x_2^\pi, z_1) + Q_\tau\Big[\beta u(x_2^\pi, x_3^\pi, z_2) + Q_\tau\Big[\beta^2 u(x_3^\pi, x_4^\pi, z_3) + \ldots \\
&\qquad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^\pi, x_{n+2}^\pi, z_n) + \cdots \Big| \cdots \Big] \cdots \Big| Z_2 = z_2 \Big| Z_1 = z\Big].
\end{aligned}
$$

It is interesting to contrast (17) or (18) with the case of expected utility. In this case, substituting $Q_\tau[\cdot]$ by $E[\cdot]$ in (18), we obtain:

$$
\begin{aligned}
V^\infty(x,z) &= u(x_1^\pi, x_2^\pi, z_1) + E\Big[\beta u(x_2^\pi, x_3^\pi, z_2) + E\Big[\beta^2 u(x_3^\pi, x_4^\pi, z_3) + \ldots \\
&\qquad \ldots + E\Big[\beta^n u(x_{n+1}^\pi, x_{n+2}^\pi, z_n) + \cdots \Big| \cdots \Big] \cdots \Big| Z_2 = z_2 \Big| Z_1 = z\Big].
\end{aligned}
$$

Using the law of iterated expectations, we can simplify this to:

$$
V^\infty(x,z) = E\left[\sum_{t=0}^{\infty} \beta^t u(x_{t+1}^\pi, x_{t+2}^\pi, z_t)\Big| Z_1 = z_1\right],
$$

which is the standard expression encountered in the economics literature. Nevertheless, one is not able to simplify (18) in this way because an analogous law of iterated expectation does not hold for quantiles, as Proposition 3.7 below shows.

We turn now to verify that this preference is dynamically consistent.

3.4 Dynamic Consistency

Our objective is to develop a dynamic theory for quantile preferences. Thus, the dynamic consistency of such preferences is of uttermost importance. In this section we formally define dynamic consistency and show that it is satisfied by the above defined preferences. The following definition comes from Maccheroni, Marinacci, and Rustichini (2006); see also Epstein and Schneider (2003).

Definition 3.6 (Dynamic Consistency). *The system of preferences* $\succeq_{t,\Omega_t}$ *is* dynamically consistent *if for every* t *and* Ω_t *and for all plans* π *and* π', $\pi_{t'}(\cdot) = \pi'_{t'}(\cdot)$ *for all* $t' \leqslant t$ *and* $\pi' \succeq_{t+1,\Omega'_{t+1},x} \pi$ *for all* Ω'_{t+1}, x, *implies* $\pi' \succeq_{t,\Omega_t,x} \pi$.

In principle, there is no reason to expect that quantile preferences would be dynamically consistent. For instance, the law of iterated expectations, which is important to the dynamic consistency of expected utility, does not have an analogous for quantile preferences, as the following result shows.

Proposition 3.7. *Let* $\Sigma_1 \supset \Sigma_0$ *be two* σ*-algebras on* Ω, $\tau \in (0,1)$, *and consider random variables* $X : \Omega \to \mathbb{R}$ *and* $Y : \Omega \to \mathbb{R}$. *Then, in general,*

$$Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] \neq Q_\tau[X|\Sigma_0]. \tag{19}$$

and it is possible that

$$Q_\tau[X|\Sigma_1]_{(\omega)} \geqslant Q_\tau[Y|\Sigma_1]_{(\omega)}, \forall \omega \in \Omega, \text{ but } Q_\tau[X|\Sigma_0]_{(\omega)} < Q_\tau[Y|\Sigma_0]_{(\omega)}, \forall \omega \in \Omega. \tag{20}$$

Note that (20) suggests a potential negation of dynamic consistency for quantile preferences in general. Fortunately, in our framework, quantile preferences are dynamically consistent and amenable to the use of the standard techniques of dynamic programming, as the following result establishes.

Theorem 3.8. *The quantile preferences defined by (9) are dynamically consistent.*

This result is important, because it implies that no money-pump can be used against a decision maker with quantile preferences. Many preferences that departure from the expected utility framework do not satisfy dynamic consistency. Indeed, Epstein and Le Breton (1993) essentially prove that dynamic consistent preferences are "probabilistic sophisticated" in the sense of Machina and Schmeidler (1992). Probabilistic sophistication *roughly* means that the preference is "based" in a probability. The result in Theorem 3.8 implicitly establishes that quantile preferences are probabilistic sophisticated. Once one understands the definitions, this does not come as a surprise, since the a quantile is just a statistics, obviously based in probability.

3.5 The Value Function

In this section we establish the existence of the value function associated to the dynamic programming problem for the quantile utility and some of its properties. This is accomplished through a contraction fixed point theorem.

The first step is to the define the contraction operator; this is similar to what we have defined in Section 3.3. For $\tau \in (0, 1]$, define the transformation $\mathbb{M}^\tau : \mathcal{C} \to \mathcal{C}$ as

$$\mathbb{M}^\tau(v)(x, z) = \sup_{y \in \Gamma(x,z)} u(x, y, z) + \beta Q_\tau[v(y, w)|z]. \tag{21}$$

Note that this is similar to the usual dynamic program problem, in which the expectation operator $E[\cdot]$ is in place of $Q_\tau[\cdot]$. The main objective is to show that the above transformation has a fixed point, which is the value function of the dynamic programming problem. The following result establishes the existence of the contraction $\mathbb{M}^\tau$ under the basic assumptions assumed throughout this paper.

Theorem 3.9. $\mathbb{M}^\tau$ *is a contraction and has a unique fixed point* $v^\tau \in \mathcal{C}$.

The unique fixed point of the problem will be the value function of the problem. Notice that the proof of this theorem is *not* just a routine application of the similar theorems from the expected utility case. In particular, the continuity of the function $(x, z) \mapsto Q_\tau[v(x, w)|z]$ is not immediate as in the standard case. Since v is not assumed to be strictly increasing in the second argument, it can be constant at some level. Constant values may potentially lead to discontinuities in the c.d.f or quantile functions; see illustration in section 7.1 in the appendix. Thus, some careful arguments are needed for establishing this continuity.

The next step is to establish the differentiability and monotonicity of the value function.

Theorem 3.10. *If Assumption 2 holds, then* $v^\tau : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in* z *and differentiable and strictly concave in an interior point* x. *Moreover,* $\frac{\partial v^\tau}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z)$, *where* $y^* \in \Gamma(x, z)$ *is the unique maximizer of (21), assumed to be interior to* $\Gamma(x, z)$.

Theorem 3.10 is the most important result in the paper, since it delivers interesting and important properties of the value function. Essentially, it establishes that the value function that one obtains from quantile functions possesses essentially the same basic properties of the value function of the corresponding expected utility problem. The second part of Theorem 3.10 is very important for the characterization of the problem. It is the extension of the standard envelope theorem for the quantile utility case. Notice that since the quantile function does not have some of the convenient properties of the expectation, we assumed that z were unidimensional (see Assumption 2) in order to establish the conclusions of Theorem 3.10. However, this unidimensionality requirement does not seem overly restrictive in most practical applications. For example, it allows us to tackle the standard asset pricing model, as section 5 shows.

Remark 3.11. *The result in Theorem 3.9 is related to that in Marinacci and Montrucchio (2010). They establish the existence and uniqueness of the value function in a more general setup. Nevertheless, we are able to provide sharper characterizations of the fixed point v^τ. In particular, Theorem 3.9 establishes that v^τ is continuous. Moreover, Theorem 3.10 shows that v^τ is differentiable, concave, and increasing.*

3.6 The Principle of Optimality

Once we have established the existence of the value function, we can show that it corresponds to the solution of the original dynamic programming problem. For this, we begin by establishing that the set of feasible plans departing from $(x,z) \in \mathcal{X} \times \mathcal{Z}$ at time t is nonempty. More formally, let us define:

$$\Pi_t(x,z) \equiv \{\pi \in \Pi(x,z) : \exists (x, z^t) \in \mathcal{X} \times \mathcal{Z}^t, \text{ with } z_t = z, \text{ such that } x_t^\pi(x, z^t) = x\}.$$

Thus, $\Pi_1(x,z)$ is just $\Pi(x,z)$. We have the following result regarding the set of feasible plans:

Lemma 3.12. *For any $x \in \mathcal{X}$ and $t \in \mathbb{N}$, $\Pi_t(x,z) \neq \varnothing$.*

This result allows us to define a supremum function as:

$$v_t^*(x,z) \equiv \sup_{\pi \in \Pi_t(x,z)} V_t(\pi, x, z). \tag{22}$$

We first observe that t plays no role in the above equation (22), that is, we prove the following:

Lemma 3.13. *For any $t \in \mathbb{N}$ and $(x,z) \in \mathcal{X} \times \mathcal{Z}$, $v_t^*(x,z) = v_1^*(x,z)$.*

Thus, we are able to drop the subscript t from (22) and write $v^*(x,z)$ instead of $v_t^*(x,z)$.

The next step is to relate v^* to v^τ, the solution of the functional equation studied in the previous section, which was proved to exist in Theorem 3.9 and satisfies the Bellman equation:

$$v^\tau(x,z) \quad = \quad \sup_{y \in \Gamma(x,z)} \{u(x,y,z) + \beta Q_\tau[v^\tau(y,w)|z]\}. \tag{23}$$

In the rest of this section we will denote v^τ simply by v.

To achieve this aim, we first establish important results relating v in equation (23) to the policy function that solves the original problem. In particular, the next result allows us to define the policy function:

Lemma 3.14. *If v is a bounded continuous function satisfying (23), then for each $(x,z) \in \mathcal{X} \times \mathcal{Z}$, the correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by*

$$\Upsilon(x,z) \equiv \{y \in \Gamma(x,z) : v(x,z) = u(x,y,z) + \beta Q_\tau[v(y,w)|z]\} \tag{24}$$

is nonempty, upper semi-continuous and therefore has a measurable selection.

Let $\psi : \mathcal{X} \times \mathcal{Z} \to \mathcal{X}$ be a measurable selection of Υ. The policy function ψ generates the plan π^{ψ} defined by $\pi_t^{\psi}(z^t) = \psi(\pi_{t-1}(z^{t-1}), z_t)$ for all $z^t \in \mathcal{Z}^t$, $t \in \mathbb{N}$.

The next result provides sufficient conditions for a solution v to the functional equation to the be supremum function, and for the plans generated by the associated policy function ψ to attain the supremum.

Theorem 3.15. *Let $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be bounded and satisfy the functional equation (23) and ψ be defined as above. Then, $v = v^*$ and the plan π^{ψ} attains the supremum in (22).*

We highlight that this generalization is not straightforward. When working with expected utility, one can employ the law of iterated expectations. However, unfortunately a similar rule does not hold for quantiles, as we have already observed in Proposition 3.7.

3.7 Euler Equation

The final step is to characterize the solutions of the problem through the Euler equation. Let $v = v^{\tau}$ be the unique fixed point of $\mathbb{M}^{\tau}$, satisfying (23). By Theorem 3.10, v is differentiable in its first coordinate, satisfying $v_{x_i}(x, z) = \frac{\partial v^{\tau}}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z) = u_{x_i}(x, y^*, z)$.

Given that we have shown the differentiability of value function, we are able to apply the standard technique to obtain the Euler equation, as formalized in the following theorem:

Theorem 3.16. *Let Assumption 2 hold. Assume that π is an optimal plan, $x_{t+1}^{\pi} \in \operatorname{int}\Gamma(x_t^{\pi}, z_t)$ and $\frac{\partial u}{\partial x_i}\left(x_t^{\pi}, x_{t+1}^{\pi}, z_t\right)$ is strictly increasing in z_t. Then, the following Euler equation holds for every $t \in \mathbb{N}$ and $i = 1, ..., p$:*

$$u_{y_i}\left(x_t^{\pi}, x_{t+1}^{\pi}, z_t\right) + \beta Q_{\tau}[u_{x_i}\left(x_{t+1}^{\pi}, x_{t+2}^{\pi}, z_{t+1}\right) | z_t] = 0. \tag{25}$$

In the expression above, u_y represents the derivative of u with respect to (some of the coordinates of) its second variable (y) and u_x represents the derivative of u with respect to (some of the coordinates of) its first variable (x).

Theorem 3.16 provides the Euler equation, that is the optimality conditions for the quantile dynamic programming problem. This result is the generalization the traditional expected utility to the quantile utility. The Euler equation in (25) is displayed as an implicit function, nevertheless for any particular application, and given utility function, one is able to solve an explicitly equation as a conditional quantile function.

The proof of Theorem relies on a result about the differentiability inside the quantile function. Indeed, for a general function h, we have $\frac{\partial}{\partial x_i} Q_{\tau}[h(x, Z)] \neq Q_{\tau}\left[\frac{\partial h}{\partial x_i}(x, Z)\right]$. However, we are able to establish this differentiability under our assumptions. We are not aware of this result in the theory of quantiles, and given its usefulness, we state it here:

Proposition 3.17. *Assume that Q_τ and $h : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ are differentiable and that h and d are increasing in z, where $d(z) \equiv h(x_i', x_{-i}, z) - h(x_i, x_{-i}, z)$ for x_i, x_i' satisfying $0 < x_i' - x_i < \epsilon$, for some small $\epsilon > 0$. Then, $\frac{\partial Q_\tau}{\partial x_i}[h(x, Z)] = Q_\tau \left[\frac{\partial h}{\partial x_i}(x, Z) \right]$.*

3.8 Example: One-Sector Growth Model

We provide a simple example to illustrate the quantile maximization utility model: the one sector-growth model (see, e.g., Brock and Mirman (1972)). We also compare the results with the corresponding model for the expected utility maximization.

Consider the one sector-growth model with the quantile maximization utility. Using the notation introduced in (18), we can write the consumer problem can be written as

$$\max_{(c_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} Q_\tau^t \left[\beta^t U(c_t) \middle| z_t \right], \tag{26}$$

subject to the following constraints:

$$\begin{aligned} c_t + k_{t+1} &= z_t h(k_t) \\ 0 &\leqslant k_{t+1} \leqslant z_t h(k_t), \end{aligned} \tag{27}$$

where c_t denotes the amount of consumption good, k_t is stock of capital, z_t is the shock, $U(\cdot)$ is the utility function, and $h(\cdot)$ is the technology.

From the results in Section 3.5, the corresponding value function for problem (26)-(27) can be expressed as

$$v(k, z) = \max_{y \in [0, z_t h(k_t)]} \left\{ U(zh(k) - y) + \beta Q_\tau \left[v(y, z') | z \right] \right\}.$$

It is easy to verify that this model satisfies Assumptions 1 and 2, and hence Theorems 3.9, 3.10, and 3.15. From Theorem 3.16, the Euler equation has the following representation:

$$-U'(z_t h(k_t) - k_{t+1}) + \beta Q_\tau \left[U'(z_{t+1} h(k_{t+1}) - k_{t+2}) z_{t+1} h'(k_{t+1}) | z_t \right] = 0.$$

By noting that $c_t = z_t h(k_t) - k_{t+1}$ and rearranging one can express the above equation as

$$Q_\tau \left[\beta(\tau) \frac{U'(c_{t+1})}{U'(c_t)} z_{t+1} h'(k_{t+1}) - 1 \middle| z_t \right] = 0. \tag{28}$$

Now we move our attention to the standard expected utility model, which can be written as

$$\max_{(c_t)_{t=0}^{\infty}} \mathrm{E} \left[\sum_{t=0}^{\infty} \beta^t U(c_t) \right], \tag{29}$$

subject to the same constraints in equation (27).

This problem can be rewritten and the associated value function is:

$$v(k,z) \quad = \max_{y\in[0,z_t h(k_t)]} \left\{ U(zh(k)-y) + \beta \mathrm{E}\left[v(y,z')|z\right]\right\}.$$

Finally, the Euler equation can be written as

$$-U'(z_t h(k_t) - k_{t+1}) + \beta \mathrm{E}\left[U'(z_{t+1}h(k_{t+1}) - k_{t+2}) z_{t+1} h'(k_{t+1})|z_t\right] = 0,$$

and by rearranging the previous equation we obtain

$$\mathrm{E}\left[\beta \frac{U'(c_{t+1})}{U'(c_t)} z_{t+1} h'(k_{t+1}) - 1 \middle| z_t\right] = 0. \tag{30}$$

When comparing the Euler equations in (28) and (30) one can notice similarities and differences. The expressions inside the conditional quantile in (28) and the conditional expectation in (30) are practically the same, except that, for the quantile model, the parameters depend on the quantile τ. That is, for each τ, we will have (potentially) different $\beta(\tau)$ and parameters of the utility function $U(\cdot)$ and technology $h(\cdot)$. Therefore, if there is relevant heterogeneity across quantiles τ, this will appear in the parameters associated to each quantile. In other words, when the parameter $\beta(\tau)$ is different from $\beta(\tau')$, one must conclude that there is relevant heterogeneity across quantiles. On the other hand, if there is no differences in the parameters across quantiles, then one can interpret this as evidence that the heterogeneity, if it exists at all, is not relevant.

This discussion suggests a central contribution of this paper: we provide a test for empirical relevance of heterogeneity. As we are going to discuss next, if one estimates the parameters and finds no variation across quantiles, one could justify the use of the expected utility framework. On the other hand, if the parameters vary across quantiles, there is evidence that the expected utility framework is not capturing relevant heterogeneity. We further discuss heterogeneity in quantile regression models in Section 4 below.

4 Estimation and Inference

In the previous section, we derived the Euler equation for the quantile utility model. For a given parametrized utility function, one is able to isolate the implicit quantile function defined by equation (25), thus obtaining the following conditional quantile model:

$$Q_\tau[m(y_t, x_t, \theta_0(\tau))|\Omega_t] = 0, \tag{31}$$

where $\tau \in (0,1)$ is a quantile of interest, (y_t, x_t) are the observable variables, Ω_t denotes the σ-field generated by $\{z_s, s \leqslant t\}$ that contains the information up to time t, and $m(\cdot)$ is a function known up to a finite dimensional vector of parameter of interest $\theta_0(\tau)$.

In this section we discuss procedures for estimation and inference of the conditional quantile functions using the generalized methods of moments (GMM).[16] More specifically, we estimate the parameters $\theta_0(\tau)$ that describe the Euler equation for each given $\tau \in (0,1)$ of interest.

It has been standard to estimate Euler equations derived from the expected utility models. It is an important exercise to learn about the structural parameters that characterize the economic problem of interest. After parametrizing the utility function, the restrictions imply a conditional average model and the parameters are commonly estimated by the GMM of Hansen (1982). We apply general (non-smooth) GMM methods to estimate general models represented by conditional moments. The methods are constructed in a manner that guarantees that the estimator is consistent and asymptotically normal, and has asymptotic covariance matrices that can be estimated consistently; hence, practical inference is simple to implement.

The model in (31) can be represented with a non-smooth conditional moment restrictions as

$$E[\tau - 1\{m(y_t, x_t, \theta_0(\tau)) < 0\}|\Omega_t] = 0. \tag{32}$$

Since $E[1\{m(y_t, x_t, \theta_0(\tau)) < 0\}|\Omega_t] = F[m(y_t, x_t, \theta_0(\tau)|\Omega_t]$, when $F(\cdot)$ is invertible, one is able to recover (31) from (32).

A general GMM estimator is defined as following. Suppose that there is a moment function vector $g(y_t, x_t, z_t, \theta_0)$ such that the population moments satisfy $E[g(y_t, x_t, z_t, \theta_0)] = 0$. A GMM estimator is the one that minimizes a square of the Euclidean distance of the sample moments from their population counterpart to zero. Let $\hat{W}$ be a positive semi-definite matrix, so that $(M'\hat{W}M)$ is a measure of distance of M from zero, and $\frac{1}{T}\sum_{t=1}^{T} g(y_t, x_t, z_t, \theta)$ is the sample analogue of its population counterpart. A GMM estimator is one that solves the following

$$\hat{\theta} = \arg\min_{\theta} \left[\frac{1}{T}\sum_{t=1}^{T} g(y_t, x_t, z_t, \theta)\right]' \hat{W} \left[\frac{1}{T}\sum_{t=1}^{T} g(y_t, x_t, z_t, \theta)\right]. \tag{33}$$

Therefore, empirical strategies for estimating the conditional quantile function in (32) lead immediately to a GMM procedure as in (33) where

$$g(y_t, x_t, z_t, \theta_0) = z_t(\tau - 1\{m(y_t, x_t, \theta_0(\tau)) < 0\}),$$

and z_t is the set of instrumental variables, such that the conditional moment in (32) is satisfied.

Given a random sample $\{(y_t, x_t, z_t) : t = 1, ...T\}$, for any given quantile τ, the parameters of interest, $\theta_0(\tau)$, can be estimated by (33). The objective function depends only on the available

[16] In a seminal paper Koenker and Bassett (1978) introduced quantile regression methods, which have been employed largely in economic applications.

sample information, the known function $m(\cdot)$, and the unknown parameters. Solutions of the above problem will be denoted by $\hat{\theta}(\tau)$, the quantile regression GMM (QR-GMM) estimator.

Identification, estimation, and inference of general (non-smooth) conditional moment restriction models as in (32) have received large attention in the econometrics literature. Identification for nonlinear semiparametric and nonparametric conditional moment restrictions models are presented in Chen, Chernozhukov, Lee, and Newey (2014). In addition, estimation and inference have been discussed by, among others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), and Chen and Liao (2015).

In a recent contribution, Chen and Liao (2015) consider a semiparametric two-step GMM for estimation and inference with weakly dependent data. Their model contains an additional nonparametric element (a vector of unknown real-valued functions) and, hence, is more general than that in equation (32), which only contains a vector of unknown finite dimensional parameters, $\theta(\tau)$. Thus, we specialize the more general results to our simpler case.[17]

Theorem 4.1 (Chen and Liao, 2015)**.** *Under standard regularity conditions, as* $T \to \infty$*, the estimator is consistent, i.e.,* $\hat{\theta} \xrightarrow{p} \theta_0$*, and*

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, V_\theta),$$

where $V_\theta = (H'WH)^{-1}(HWV_1WH)(H'WH)^{-1}$*,* $V_1 = Avar\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g(y_t, x_t, z_t, \theta_0)\right)$*, and* $H = \frac{\partial}{\partial\theta}E[g(y_t, x_t, z_t, \theta)]$.

Given the result in Chen and Liao (2015) (Theorem 4.1), one is able to estimate the variance-covariance matrix and conduct practical inference for the parameters of interest.

A few key observations should be noted. First, for a given random sample $\{(y_t, x_t, z_t) : t = 1, ...T\}$, we are able to apply the QR-GMM methods and estimate the parameters $\theta(\tau)$ across different quantiles τ. Second, for any given example, applying the QR-GMM requires specifying the function $m(\cdot)$, the observable variables (y_t, x_t), and the information set Ω_t, and hence, the instruments z_t. Note that in a very simple case $z_t = x_t$. The instruments are used to achieve a valid orthogonality condition for the Euler equation, that is, the (conditional) moment condition equals to zero. The idea is that by imposing certainty equivalence on the nonlinear rational expectations model, the instruments help to circumvent some of the difficulties in obtaining a complete characterization of the stochastic equilibrium.[18] Third, we can allow for conditional heteroskedasticity and can conduct statistical inference without explicitly characterizing the dependence of the conditional variances on the information set. In the context of the asset pricing models discussed in Section 5 below, for example, this feature

[17] This result is also given in Theorem 7.2 in Newey and McFadden (1994).

[18] In the literature, it is standard to estimate Euler equations for conditional average models by parametrizing the utility function and estimating the parameters of interest using instrumental variables GMM (Hansen (1982)).

of our estimation procedure allows the conditional variances of asset yields to fluctuate with movements of variables in the conditioning information set.

An interpretation for this procedure is that the τ-agent maximizes the τ-th quantile, and the econometrician then wishes to learn about the potential underlying heterogeneity across (agents) quantiles. We note that the theoretical methods do not impose restrictions across the quantiles, and, thus, empirically the parameter estimates, as a function of the quantiles, might (or might not) reveal the underlying heterogeneity. In fact, QR models are an important tool to capture heterogeneity in applications. To gain intuition, we consider simple examples from linear models with and without heterogeneity. Assume that y_t is given by the following linear model:

$$y_t = \beta_1 + \beta_2 x_t + u_t, \tag{34}$$

where u_t is an unobservable error term.

First, consider the case in which the data $\{y_t, x_t\}_{t=1}^{T}$ are independent and identically distributed (i.i.d.). When using the conditional expectation, the innovation term is assumed to satisfy the following condition $E[u_t|x_t] = 0$. Thus, by applying the conditional expectation operator on both sides of (34) and using the conditional mean zero, we obtain:

$$E[y_t|x_t] = \beta_1 + \beta_2 x_t. \tag{35}$$

Therefore, one can estimate the parameters (β_1, β_2) by employing standard OLS or GMM techniques, thus obtaining estimates $(\hat{\beta}_1, \hat{\beta}_2)$. Notice that (35) gives only the conditional expectation of y_t given x_t. From using it, one forgoes the possibility of learning about the whole distribution of y_t given x_t.

In this simple i.i.d. case, the quantile functions are simply a vertical displacement of one another with population parameters $(\beta_1 + F_\tau^{-1}, \beta_2) = (\beta_1(\tau), \beta_2)$. To see this, take the conditional quantile function on both sides of (34)

$$\begin{aligned}
Q_\tau[y_t|x_t] &= Q_\tau[\beta_1 + \beta_2 x_t + u_t|x_t] \\
&= \beta_1 + \beta_2 x_t + F_\tau^{-1}[u_t|x_t] \\
&= (\beta_1 + F_\tau^{-1}[u_t|x_t]) + \beta_2 x_t \\
&= \beta_1(\tau) + \beta_2 x_t.
\end{aligned}$$

This model allows only for a location shift, where $(\beta_1(\tau), \beta_2)$ and only β_1 depends on the quantile τ. Given the restriction $F_\tau^{-1}[u_t|x_t] = 0$, one applies QR to the above model to obtain the estimates $(\hat{\beta}_1(\tau), \hat{\beta}_2(\tau))$. Notice that in this simple i.i.d. case there is absence of heterogeneity and the estimates of $\hat{\beta}_2$ should not depend on the quantile τ. Hence, in the nonexistence of heterogeneity, the model for the conditional average would estimate the same β_2 as that for the QR.

In contrast, consider the heterogeneous case with heteroskedasticity as

$$y_t = \beta_1 + \beta_2 x_t + \sigma(x_t)\varepsilon_t, \tag{36}$$

where $u_t \equiv \sigma(x_t)\varepsilon_t$ and $\sigma(x_t) = (1 + \gamma x_t)$, for example. Taking the conditional quantile functions of y_t in (36)

$$\begin{aligned} Q_\tau[y_t|x_t] &= \beta_1 + \beta_2 x_t + \sigma(x_t)F_\tau^{-1}[\varepsilon_t|x_t] \\ &= (\beta_1 + F_\tau^{-1}[\varepsilon_t|x_t]) + (\beta_2 + \gamma F_\tau^{-1}[\varepsilon_t|x_t])x_t \\ &= \beta_1(\tau) + \beta_2(\tau)x_t. \end{aligned}$$

In this case both parameters $(\beta_1(\tau), \beta_2(\tau))$ that describe the quantile functions depend on the quantile τ. Thus, the QR $(\hat\beta_1(\tau), \hat\beta_2(\tau))$ are estimates for the population parameters $(\beta_1 + F_\tau^{-1}, \beta_2 + \gamma F_\tau^{-1}) = (\beta_1(\tau), \beta_2(\tau))$. Therefore, when there is heterogeneity in the data, the estimates of both coefficients should depend on the quantile τ. Finally, in the presence of heterogeneity, the model for the conditional average would estimate β_2 and, hence, not capture any heterogeneity.

Remark 4.2. *In this paper, we are interested in estimating the conditional quantile functions to learn about the underlying heterogeneity among agents. Nevertheless, it is possible to see the quantile τ as a parameter to be estimated together with the parameters $\theta_0(\tau)$. Bera, Galvao, Montes-Rojas, and Park (2016) develop an approach that delivers estimates for the coefficients together with a representative quantile. In their framework, τ captures a measure of asymmetry of the conditional distribution of interest and is associated with the "most probable" quantile in the sense that it maximizes the entropy.*

5 Application: Asset Pricing Model

This section illustrates the usefulness of the new quantile utility maximization methods through an empirical example. We apply the methodology to the standard asset-pricing model, which is central to contemporary economics and finance. It has been used extensively in the literature and has had remarkable success in providing empirical estimates for the study of the risk-aversion and discount-factor parameters. We refer the readers to Cochrane (2005), Mehra (2008), and Ljungqvist and Sargent (2012), and the references therein, for a comprehensive overview.

We employ a variation of Lucas (1978)'s endowment economy (see, also, Hansen and Singleton (1982), Mehra and Prescott (1985) and Mehra and Prescott (2008)). The economic agent decides on the intertemporal consumption and savings (assets to hold) over an infinity horizon economy, subject to a linear budget constraint. The decision generates an intertemporal policy function, which is used to estimate the parameters of interest for a given utility function.

Let c_t denote the amount of consumption good that the individual consumes in period t. At the beginning of period t, the consumer has x_t units of the risky asset, which pays dividend z_t. The price of the consumption good is normalized to one, while the price of the risky asset in period t is $p(z_t)$. Then, the consumer decides how many units of the risky asset x_{t+1} to save for the next period, and its consumption c_t, satisfying:

$$c_t + p(z_t)x_{t+1} \leqslant [z_t + p(z_t)] \cdot x_t, \tag{37}$$

$$c_t, x_{t+1} \geqslant 0. \tag{38}$$

Using the notation introduced in (18), we can write the consumer problem as the maximization of

$$\sum_{t=0}^{\infty} Q_\tau^t \left[\beta^t U(c_t) \middle| \Omega_t \right], \tag{39}$$

subjected to (37) and (38), where $\beta \in (0,1)$ is the discount factor, and $U : \mathbb{R}_+ \to \mathbb{R}$ is the utility function.

Because we have a single agent, the holdings must not exceed one unit. In fact, in equilibrium, we must have $x^*_{tk} = 1, \forall t, k$. Let $\bar{x} > 1$ and $\mathcal{X} = [0, \bar{x}]$.

From (37), we can determine the consumption entirely from the current and future states, that is, $c_t = z_t \cdot x_t + p(z_t) \cdot (x_t - x_{t+1})$. Following the notation of the previous sections, we denote x_t by x, x_{t+1} by y, and z_t by z. Then, the above restrictions are captured by the feasible correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{X} = \mathcal{X}$ defined by:

$$\Gamma(x,z) \equiv \{ y \in \mathcal{X} : p(z) \cdot y \leqslant (z + p(z)) \cdot x \}. \tag{40}$$

For each pricing function $p : \mathcal{Z} \to \mathbb{R}_+$, define the utility function as:

$$u(x,y,z) \equiv U\left[z \cdot x + p(z) \cdot (x - y) \right]. \tag{41}$$

We assume the following:

Assumption 3. *(i)* $\mathcal{Z} \subseteq \mathbb{R}$ *is a bounded interval and* $\mathcal{X} = [0, \bar{x}]$*;*

(ii) $U : \mathbb{R}_+ \to \mathbb{R}$ *is given by* $U(c) = \frac{1}{1-\gamma} c^{1-\gamma}$*, for* $\gamma > 0$*;*

(iii) z *follows a Markov process with pdf* $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$*, which is continuous, symmetric,* $f(z,w) > 0$*, for all* $(z,w) \in \mathcal{Z} \times \mathcal{Z}$ *and satisfies the property: if* $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leqslant z'$*, then:*

$$\int_{\mathcal{Z}} h(\alpha) f(\alpha|z) d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha) f(\alpha|z') d\alpha; \tag{42}$$

(iv) $z \mapsto z + p(z)$ *is* C^1 *and non-decreasing, with* $z\left(\ln(z + p(z))\right)' \geqslant \gamma$*.*

Assumptions 3$(i)-(ii)$ are standard in economic applications. Assumption 3(iii) means that a high value of the dividend today makes a high value tomorrow more likely. It implies Assumption 2(ii). Assumption 3(iv), $z \mapsto z + p(z)$ is non-decreasing, is natural. It states that the price of the risky asset and its return are a non-decreasing function of the dividends. Note that it is natural to expect that the price is non-decreasing with the dividends, but Assumption 3(iv) is even weaker than this, as it allows the price to decrease with the dividend; only $z+p(z)$ is required to be non-decreasing.[19]

Given Assumption 3, we can verify the assumptions for establishing the quantile utility model in the asset pricing model context. Thus, we have the following:

Lemma 5.1. *Assumption 3 implies Assumptions 1 and 2 and Theorem 3.16 holds.*

Therefore, Theorems 3.9 and 3.10 imply the existence of a value function v, which is strictly concave and differentiable in its first variable, satisfying

$$v(x,z) = \max_{y \in \Gamma(x,z)} Q_\tau[g(x,y,z,\cdot)|z],$$

where

$$g(x,y,z,w) = u\,(x,y,z) + \beta v(y,w).$$

Also, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$. Note that

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y,z) &= U'\left[z \cdot x + p(z)\cdot(x-y)\right](z+p(z))\,;\\ \frac{\partial u}{\partial y}(x,y,z) &= U'\left[z \cdot x + p(z)\cdot(x-y)\right](-p(z))\,;\end{aligned}$$

Because, in equilibrium, the holdings are $x_t = 1$ for all t, we can derive the Euler equation as in (25) for this particular problem to obtain:

$$-p(z_t)U'\left(c_t\right) + \beta Q_\tau[U'\left(c_{t+1}\right)\left(z_{t+1} + p(z_{t+1})\right)|\Omega_t] \quad = \quad 0. \tag{43}$$

Let us define the return by:

$$1 + r_{t+1} \equiv \frac{z_{t+1} + p(z_{t+1})}{p(z_t)}. \tag{44}$$

Therefore, the Euler equation in (43) simplifies to:

$$Q_\tau\left[\beta(\tau)(1+r_{t+1})\frac{U'(c_{t+1})}{U'(c_t)}\middle|\Omega_t\right] = 1. \tag{45}$$

[19]In our dataset, when regressing the returns on the dividends, we find a statistically positive correlation.

The Euler equation in (45) possesses a nonlinear conditional quantile function representation as in (31). Thus, for a given utility function, one is able to estimate the parameters of interest using the quantile regression GMM (QR-GMM) methods described in Section 4 above.

5.1 Estimation

In this application we follow a large body of the literature, as for example, Hansen and Singleton (1982) and Stock and Wright (2000), among others, and use a constant relative risk aversion (CRRA) utility function as

$$U(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma},$$

for $\gamma > 0$. The parameter γ is the standard measure the degree of relative risk aversion that is implicit in the utility function.

The ratio of marginal utilities can be written as

$$\frac{U'(c_{t+1})}{U'(c_t)} = \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}. \tag{46}$$

Finally, from equations (45) and (46), the Euler equation can be rewritten as

$$Q_\tau \left[\beta(\tau)(1 + r_{t+1}) \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma(\tau)} - 1 \middle| \Omega_t \right] = 0. \tag{47}$$

After deriving the Euler equation in (47), we aim to estimate the parameters of interest $(\gamma(\tau), \beta(\tau))$. Given a random sample $\{(r_t, c_t) : t = 1, ...T\}$, we are able to apply the QR-GMM methods and, for each quantile $\tau \in (0, 1)$, estimate the corresponding parameters $(\gamma(\tau), \beta(\tau))$. In this way, we uncover the potential underlying heterogeneity across the quantiles.

Notice that there are two measures of riskiness in this model. First, for a fixed quantile τ, $\gamma(\tau)$ captures the relative risk aversion, for which a larger $\gamma(\tau)$ signifies a larger risk aversion. Second, the model also captures the risk across quantiles. Theorem 2.1, in Section 2.3 above, predicts that the agent that maximizes the larger quantile is more risk taker; thus, the theorem suggests that the coefficient of relative risk aversion should decrease over the quantiles, that is $\gamma(\tau') < \gamma(\tau)$ for $\tau' > \tau$.

Several considerations are in order when estimating the parameters in (47). First, equation (47) is an equilibrium condition. This is commonly used in the literature to derive orthogonality conditions based on instrumental variables that can be used to estimate the parameters of the utility function. In this paper, for simplicity, we abstract from the instruments, as a first approach to the problem, and estimate the parameters with a nonlinear quantile regression model of the excess of returns on the ratio of consumption.[20] Second, when bringing (47) to

[20] In spite of that, as we will see below, our estimates for average models are very close to those in the existing literature.

the data, rational expectations is an underlying assumption. This means that the conditional quantile function operator in (47) coincides with the theoretical given all information available to the consumer at time t. Thus, the conditional quantile function is valid over time. Third, we abstract from the presence of the "taste-shock" (or measurement error). All of these assumptions and simplifications have been largely discussed in terms of models for estimating conditional averages (see, e.g., Attanasio and Low (2004)). Because the main objective of this paper is to provide a first view of the quantile utility maximization problem, we use these assumptions for simplicity. Nevertheless, extending the methods to relax these assumptions, for example, considering instrumental variables and measurement errors, is an important direction for future research. Finally, we restrict the discount factor coefficients to satisfy $\beta(\tau) < 1$ for all quantiles and estimate both parameters of interest simultaneously.

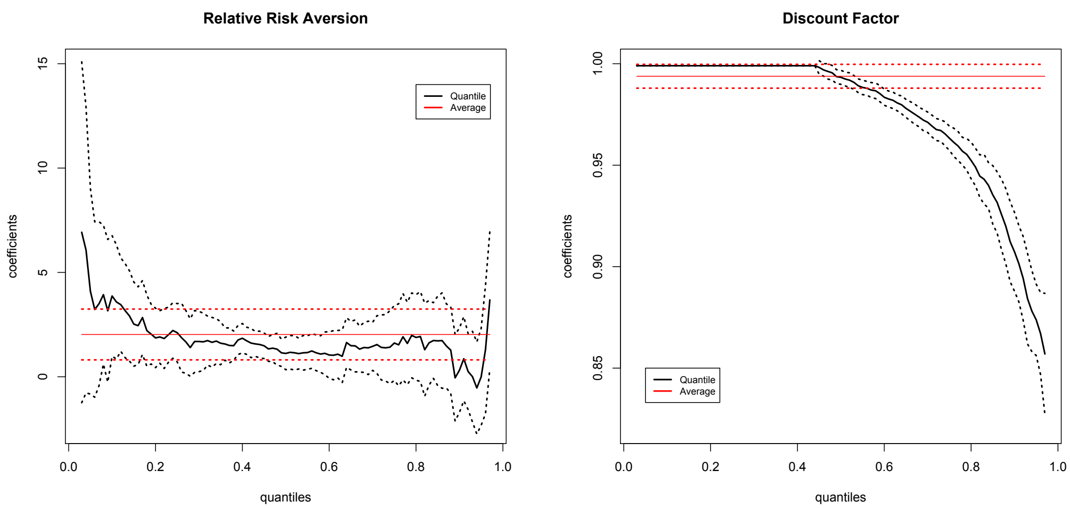
It is also important to note that, recently, there has been an attempt to allow for heterogeneity in dynamic nonlinear rational expectation models, especially in the context of estimating the coefficient of risk aversion. A class of models allows for heterogeneity across agents in lieu of a single representative agent. Nevertheless, although the individuals are heterogeneous, it is usual to impose the assumption of homogeneous parameters across individuals; that is, the coefficient of risk aversion is common across individuals. This former condition has become standard in the literature (see, e.g, Heaton and Lucas (2008)). As an alternative, Mazzocco (2008) develops a two-agent model wherein the risk-aversion is allowed to vary for both agents. Moreover, Herranz, Krasa, and Villamil (2015) capture heterogeneity in risk-aversion by imposing distributional assumptions on the coefficient of risk aversion; in particular, they assume that it is normally distributed. However, all of these constraints might be seen as excessively strong and this paper moves in the direction of relaxing them. We develop a complement and alternative to these models by considering the quantile utility maximization, whereby we are able to estimate the parameters indexed by the corresponding quantiles.

5.2 Data

We use a data set that is common in the literature for modeling stock prices, as discussed in the previous section. We use monthly data from 1959:01 to 2015:11, which produces 683 observations. As is standard in the literature (see, e.g., Hansen and Singleton (1982)), two different measures of consumption were considered: nondurables, and nondurables plus services. The monthly, seasonally adjusted observations of aggregate nominal consumption (measured in billions of dollars unit) of nondurables and services were obtained from the Federal Reserve Economic Data. Real per capita consumption series were constructed by dividing each observation of these series by the corresponding observation of population, deflated by the corresponding CPI (base 1973:01).

Each measure of consumption was paired with four sets of stock returns from the Center for Research in Security Prices (CRSP) U.S. Stock database, which contains month-end prices for

Figure 1: Nondurables plus services and VWR without dividends



primary listings for the New York Stock Exchange (NYSE). We use the value-weighted average return (VWR) (including dividends or excluding dividends) on all stocks listed on the NYSE. In addition, we employ the equally-weighted average of returns (EWR) (including dividends or excluding dividends) on the NYSE. The nominal returns were converted to real returns by dividing by the deflator associated with the measure of consumptions.

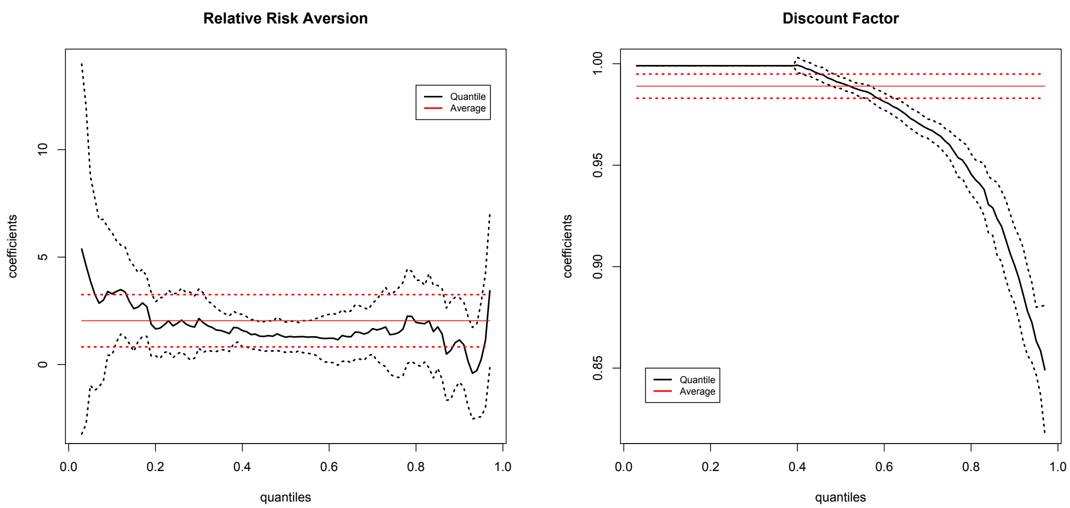
5.3 Results

Now we present the results. Because the literature reports results for conditional mean models, for comparison purposes, we also estimate the standard conditional expectation regression GMM (ER-GMM) version of the model.

The results for the estimates of the parameters of interest are reported in Figures 1–4. The panels on the left display the relative risk aversion coefficients, and the panels on the right, the discount factor. We present estimates using both consumption of nondurables, and nondurables plus services. We also provide the results for stock return VWR, with and without dividends. For brevity, we omit the results for the stock return EWR; nevertheless, the results are very similar. In addition, the figures present results for the coefficients and confidence bands, for a range of quantiles, for QR-GMM and ER-GMM (straight red lines), respectively. The dashed region in each panel represents the 95% confidence interval.

Figure 1 presents standard QR-GMM and ER-GMM estimates of the coefficient of relative risk-aversion γ and discount factor β, using consumption of nondurables plus services and stock return VWR without dividends for the relative risk-aversion estimates. The plot on the left displays the relative risk-aversion estimates. The first interesting observation is that the results document strong evidence of heterogeneity in the coefficient of the risk-aversion

Figure 2: Nondurables plus services and VWR with dividends

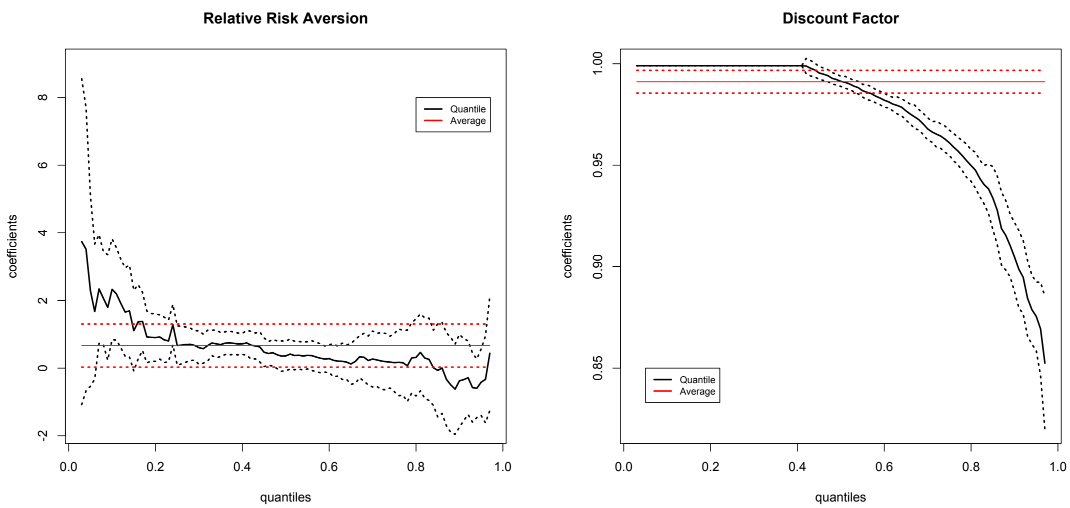


factor across quantiles. In particular, Figure 1 shows that the coefficient of risk aversion is relatively larger for lower quantiles, achieving values around 7. In addition, the coefficient of risk aversion decreases when the quantile index increases. This result is consistent with the result in Theorem 2.1 (Manski (1988)), which shows that the agent that maximizes the higher quantile is the more risk-loving. We find risk aversion estimates of 1.45 for the median, which is in line with literature, for example Herranz, Krasa, and Villamil (2015) (1.5–1.56) and Mazzocco (2008) (1.7 for men) and parameter values used in the real business-cycle literature. Regarding the conditional average result, the ER-GMM estimate is 2, with a standard error of 0.73. Using GMM methods, Hansen and Singleton (1982) estimate the coefficient of relative risk-aversion between 0.05 and 0.32. More recently, Stock and Wright (2000) present a 90% a confidence interval for the coefficient of risk aversion ranging from −2.0 to 2.3. Thus, the overall results obtained here for conditional median and average models are relatively close to those in the literature. Nevertheless, different from the literature, our work is able to uncover strong heterogeneity, especially at the tails.

The estimates for the discount factor, in the right plot of Figure 1, also are interesting. First, the figure shows that, for low quantiles, the discount factor reaches the boundary and, hence, is close to unity. Second, the figure shows heterogeneity in the discount factor parameter for the upper quantiles. Overall, Figure 1 shows evidence that the discount factor is larger for lower quantiles. That is, the more risk averse the agents, the more patient they are.

Remark 5.2. *It should be noted that our model does not control for income or wealth. Thus, the agents that correspond to low quantiles* do not *necessarily correspond to low income, but to low risk aversion. This observation is important to avoid confusion with the results in a branch of the literature that links discount factors with income and wealth (see, e.g., Hausman (1979),*

Figure 3: Nondurables and VWR without dividends

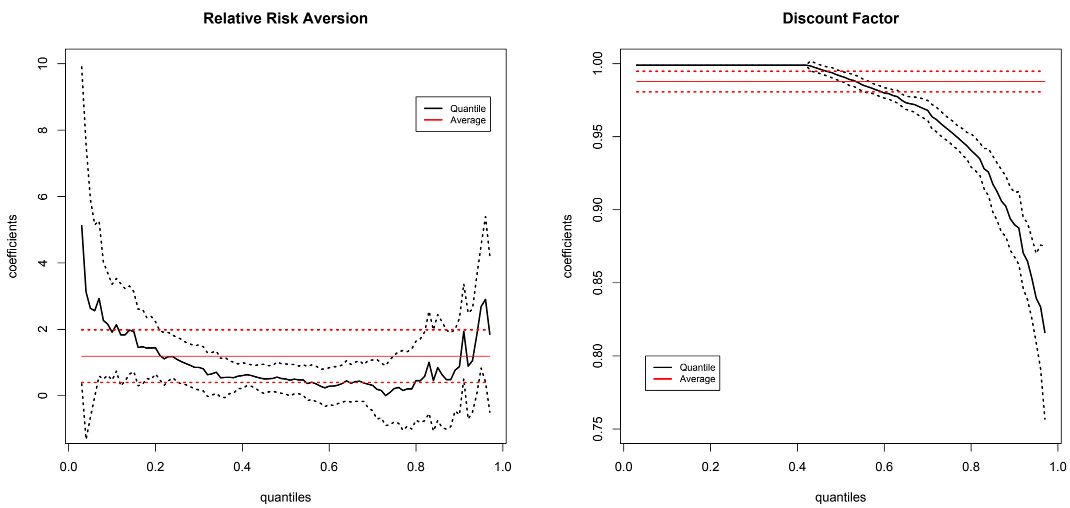


and [*Lawrance*](#) *(*[*1991*](#)*)). Moreover, there is empirical evidence that documents small discount factors estimates. This literature estimates discount factors by using a quasi-hyperbolic discount function (see, e.g.,* *Paserman (2008), Fang and Silverman (2009), and Laibson, Maxted, Repetto, and Tobacman (2015)). In contrast to these streams of literature, this paper abstracts from a relationship between discount rates and poverty and employs a simple model to estimate the discount factor. Our objective is to illustrate the potential empirical application of the quantile utility maximization model. We leave the connection with income and wealth and related extensions for future research.*

The results for the estimates of the coefficient of risk aversion in Figure 1 also might help to shed light on the equity premium puzzle (Mehra and Prescott (1985)). As Mehra and Prescott (2008) state, the standard theory is consistent with the notion of risk that, on average, stocks should return more than bonds. The puzzle arises from the fact that the quantitative predictions of theory are an order of magnitude different from what has been historically documented; that is, the coefficient of risk aversion required would need to be much larger than the one observed from estimated models. Nevertheless, the results from the QR-GMM estimates of the quantile utility model document strong heterogeneity across quantiles. Thus, if one interprets different quantiles as different individuals, there is strong heterogeneity for the coefficient of risk-aversion and, hence, the discrepancy on returns could be rationalized with different coefficients of risk aversion among economic agents. Thus, these results make it possible to reconcile the relative large spread observed between the risk-free and risky assets with the large relative risk aversion of an individual who is solving the intertemporal optimization problem for lower quantiles.

Figures 2–4 serve as robustness checks and display the estimates for our proposed methods.

Figure 4: Nondurables and VWR with dividends



The results are qualitatively similar to those in Figure 1. The coefficients of relative risk-aversion, in the panels in the left of the figures, present strong heterogeneity over the quantiles. The relative risk aversion decreases across quantiles, being larger for lower quantiles and smaller for upper quantiles. This result is consistent with the theoretical predictions in terms of risk when using the quantile utility maximization model. The results suggest that agents behave in a more risk averse manner for the lower part of the conditional distribution of stock returns. In contrast, for upper quantiles, the risk aversion is smaller, providing evidence that agents behave in a more risk-loving manner. In addition, the discount factor estimates, in the panels in the right of the figures, present heterogeneity across quantiles, especially for upper quantiles. The discount factor is smaller for more risk-taking agents, which suggests that those agents are less patient. On the contrary, for lower quantiles, the risk aversion is large, as is the discount factor, providing evidence that more risk-averse agents are more patient.

From the Figures 1–4, we note that all of the QR estimates show the same pattern; that is, both the relative risk aversion and discount factor decrease as a function of the quantiles. The variability of the effects is the most apparent and dramatic in the tails of the conditional distribution of returns, whereas the ER-GMM point estimates are relatively smaller.

In all, the application illustrates that the new methods serve as an important tool to study economic behavior, in particular, asset pricing. The methods allow one to uncover heterogeneity by estimating the relative risk aversion and discount factor at different quantiles, which might be viewed as reflecting the different risk behavior of agents. Our empirical results document heterogeneity of risk-aversion and discount factor, providing empirical evidence that it would be possible to reconcile the equity premium puzzle when investigating the entire distribution instead of concentrating only on the mean.

6 Summary and Open Questions

This paper develops a dynamic model of rational behavior under uncertainty for an agent maximizing the quantile utility function indexed by a quantile $\tau \in (0,1]$. More specifically, an agent maximizes the stream of future τ-quantile utilities, where the quantile preferences induce the quantile utility function. We show dynamic consistency of the preferences and that this dynamic problem yields a value function, using a fixed-point argument. We also obtain desirable properties of the value function. In addition, we derive the corresponding Euler equation. The quantile utility maximization model allows us to account for heterogeneity through the quantiles.

Empirically, we show that one can employ existing general (non-smooth) generalized method of moments methods for estimating and testing the rational quantile models directly from stochastic Euler equations. An attractive feature of this method is that the parameters of the dynamic objective functions of economic agents can be interpreted as structural objects. Finally, to illustrate the methods, we construct an asset-pricing model and estimate the implied risk aversion and discount factor parameters. The results suggest evidence of heterogeneity in both parameters, as both risk aversion and discount factor decrease as a function of the quantiles.

Many issues remain to be investigated. The extension of the quantile maximization model from considering a single quantile to multiple quantiles simultaneously would be important. Extensions of the methods to general equilibrium models pose challenging new questions. In addition, aggregation of the quantile preferences is also a critical direction for future research. Applications to asset pricing and consumption models would appear to be a natural direction for further development of quantile utility maximization models.

7 Appendix

7.1 Properties of Quantiles

The following picture illustrates the c.d.f. F of a random variable X, and its corresponding quantile function $Q(\tau) = \inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant \tau\}$, for $\tau > 0$.[21] In this case, X assumes the value 3 with 50% probability and is uniform in $[1,2] \cup [4,5]$ with 50% probability. This picture is useful to inspire some of the properties that we state below. Note, for instance, the discontinuities and the values over which the quantile is constant.

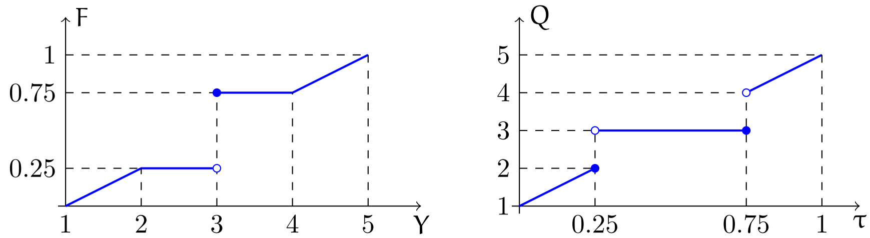


Figure 5: c.d.f. and quantile function of a random variable.

The following lemma is an auxiliary result that will be helpful for the derivations below.

Lemma 7.1. *The following statements are true:*

(i) Q *is increasing, that is,* $\tau \leqslant \hat{\tau} \implies Q(\tau) \leqslant Q(\hat{\tau})$.

(ii) $\lim_{\tau \downarrow \hat{\tau}} Q(\tau) \geqslant Q(\hat{\tau})$.

(iii) Q *is left-continuous, that is,* $\lim_{\tau \uparrow \hat{\tau}} Q(\tau) = Q(\hat{\tau})$.

(iv) $\Pr(\{z : z < Q(\tau)\}) \leqslant \tau \leqslant \Pr(\{z : z \leqslant Q(\tau)\}) = F(Q(\tau))$.

(v) If $g : \mathbb{R} \to \mathbb{R}$ *is a continuous and strictly increasing function, then* $Q_\tau[g(X)] = g(Q_\tau[X])$.

(vi) If $g, h : \mathbb{R} \to \mathbb{R}$ *are such that* $g(\alpha) \leqslant h(\alpha), \forall \alpha$, *then* $Q_\tau[g(Z)] \leqslant Q_\tau[h(Z)]$.

(vii) F *is continuous if and only if* Q *is strictly increasing.*

(viii) F *is strictly increasing if and only if* Q *is continuous.*

Proof. (i) Let us first assume $\tau > 0$. If $\tau \leqslant \hat{\tau}$, then $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \tau\} \supseteq \{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \hat{\tau}\}$. This implies $Q_Z(\tau) \leqslant Q_Z(\hat{\tau})$. Next, if $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = -\infty$, there is nothing else to prove. If $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = x \in \mathbb{R}$, then $F_Z(x - \epsilon) = 0$ for any $\epsilon > 0$. Let $\hat{\tau} > 0$. Then, $y \in \{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \hat{\tau}\} \implies y > x - \epsilon$, which in turn implies $Q_Z(\hat{\tau}) \geqslant x - \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $Q_Z(\hat{\tau}) \geqslant x = Q_Z(0)$, which concludes the proof.

(ii) From (i), $\lim_{\tau \downarrow \hat{\tau}} Q_Z(\tau) \geqslant \inf_{\tau \geqslant \hat{\tau}} Q_z(\tau) \geqslant Q_z(\hat{\tau})$. Figure 5 illustrates (for example for $\hat{\tau} = 0.25$) that the inequality can be strict.

(iii) From (i), we know that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) \leqslant Q_z(\hat{\tau})$. For the other inequality, assume that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) + 2\epsilon < Q_z(\hat{\tau}) < \infty$, for some $\epsilon > 0$. This means that for each $k \in \mathbb{N}$, we can find

[21] For $\tau = 0$, $Q(0) = \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}$ is just the lower limit of the support of the variable.

$\alpha^k \in \{\alpha : F_Z(\alpha) \geqslant \hat{\tau} - \frac{1}{k}\}$ such that $Q_Z(\hat{\tau} - \frac{1}{k}) \leqslant \alpha^k \leqslant Q_Z(\hat{\tau} - \frac{1}{k}) + \epsilon < Q_Z(\hat{\tau}) - \epsilon$. We may assume that $\{\alpha^k\}$ is an increasing sequence bounded by $Q_z(\hat{\tau})$ and thus converges to some $\bar{\alpha} \in \mathbb{R}$. Then, $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) \leqslant \bar{\alpha} \leqslant Q_z(\hat{\tau}) - \epsilon < Q_z(\hat{\tau})$. Since $F_Z(\alpha^k) \geqslant \hat{\tau} - \frac{1}{k}$ and F_Z is upper semi-continuous, $F_Z(\bar{\alpha}) \geqslant \hat{\tau}$, which implies that $\bar{\alpha} \geqslant Q_Z(\hat{\tau})$, a contradiction. Now, assume that $Q_Z(\hat{\tau}) = \infty$. Since $\lim_{\alpha \to \infty} F_Z(\alpha) = 1$, the set $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \tau\}$ is non-empty for all $\tau < 1$, that is, $Q_Z(\tau) < \infty$ for all $\tau < 1$. Thus, $\hat{\tau} = 1$. If $\lim_{\tau \uparrow 1} Q_Z(\tau) = x \in \mathbb{R}$, then $F_Z(x+1) \geqslant 1 - \epsilon$ for all $\epsilon > 0$, which implies that $F_Z(x+1) = 1$ and $Q_Z(1) \leqslant x + 1$, a contradiction.

(iv) As above, if $Q_Z(\tau) = \infty$, then $\tau = 1$, which implies $1 = \Pr(\{w : z < \infty\}) = \Pr(\{w : z \leqslant \infty\})$ and there is nothing to prove. Let $\bar{\alpha} = Q_Z(\tau) < \infty$. If $\alpha^k \downarrow \bar{\alpha}$ is such that $F_Z(\alpha^k) \geqslant \tau$, then $F_Z(\bar{\alpha}) \geqslant \tau$, by the well-known upper-semicontinuity of F_Z. That is, $\tau \leqslant F_Z(Q_Z(\tau))$. For the other inequality, let $\alpha^k \uparrow \bar{\alpha} = Q_Z(\tau)$. Since $\alpha^k < \bar{\alpha}$, then $\Pr[Z \leqslant \alpha^k] < \tau$, by the definition of $\bar{\alpha}$. Thus, $\Pr[Z < \alpha^k] \leqslant \Pr[Z \leqslant \alpha^k] < \tau$ and $\Pr[Z < \bar{\alpha}] \leqslant \sup_k \Pr[Z < \alpha^k] \leqslant \tau$.

(v) The proof is direct as follows:

$$\begin{aligned}
Q_\tau(g(Z)) &= \inf\{\alpha \in \mathbb{R} : \Pr[g(Z) \leqslant \alpha] \geqslant \tau\} \\
&= \inf\{\alpha \in \mathbb{R} : \Pr\left[Z \leqslant g^{-1}(\alpha)\right] \geqslant \tau\} \\
&= \inf\{\alpha \in \mathbb{R} : g^{-1}(\alpha) = \beta, \Pr[Z \leqslant \beta] \geqslant \tau\} \\
&= \inf\{g(\beta) : \Pr[Z \leqslant \beta] \geqslant \tau\} \\
&= g(\inf\{\beta : \Pr[Z \leqslant \beta] \geqslant \tau\}) \\
&= g(Q_\tau(Z)).
\end{aligned}$$

(vi) Since $g \leqslant h$, then for any α, $\{z : g(z) \leqslant \alpha\} \supseteq \{z : h(z) \leqslant \alpha\}$, which implies $F_{g(Z)}(\alpha) = \Pr[g(Z) \leqslant \alpha] \geqslant \Pr[h(Z) \leqslant \alpha] = F_{h(Z)}(\alpha)$. If $\tau > 0$, $\{\alpha \in \mathbb{R} : \Pr[g(Z) \leqslant \alpha] \geqslant \tau\} \supseteq \{\alpha \in \mathbb{R} : \Pr[h(Z) \leqslant \alpha] \geqslant \hat{\tau}\}$. Taking infima, we obtain $Q_{g(Z)}(\tau) \leqslant Q_{h(Z)}(\tau)$. On the other hand, $\{\alpha \in \mathbb{R} : F_{h(Z)}(\alpha) = 0\} \subset \{\alpha \in \mathbb{R} : F_{g(Z)}(\alpha) = 0\}$ and taking the supremum in both sides we obtain the same conclusion.

(vii) Assume that F_Z is discontinuous at x_0, that is, $\lim_{x \uparrow x_0} F_Z(x) = y_0 < y_1 = F_Z(x_0)$. If $y_0 < y_2 < y_3 < y_1$, then $Q_Z(y_2) = \inf\{\alpha : F_Z(\alpha) \geqslant y_2\} = \inf\{\alpha : F_Z(\alpha) \geqslant y_3\} = Q_Z(y_3)$, that is, Q_Z is not strictly increasing. Conversely, assume that Q_Z is not strictly increasing, that is, there exists $y_2 < y_3$ such that $Q_Z(y_2) = Q_Z(y_3) = x$. By definition, this means that $F_Z(x - \epsilon) < y_2 < y_3 \leqslant F_Z(x + \epsilon)$, for all $\epsilon > 0$. But this implies that F_Z is not continuous at x.

$(viii)$ Suppose that F_Z is not strictly increasing, that is, there exists $x_1 < x_2$ such that $F_Z(x_1) = F_Z(x_2) = y$. Then, $Q_Z(y - \epsilon) = \inf\{\alpha : F_Z(\alpha) \geqslant y - \epsilon\} \leqslant x_1 < x_2 \leqslant \inf\{\alpha : F_Z(\alpha) \geqslant y + \epsilon\} = Q_Z(y + \epsilon)$. Thus, Q_Z cannot be continuous at y. Conversely, assume that Q_Z is not continuous at y_0. Since Q_Z is increasing by (i) and left-continuous by (iii), this means that $Q_Z(y_0) = x_0 < x_1 = \lim_{y \downarrow y_0} Q_Z(y)$. If $x_0 < x_2 < x_1$, then $F_Z(x_2) \leqslant y_0$, otherwise $\lim_{y \downarrow y_0} Q_Z(y) \leqslant x_2$. By (iv), we have $y_0 \leqslant F_Z(Q_Z(y_0)) = F_Z(x_0) \leqslant F_Z(x_2) \leqslant y_0$, that is, F_Z is not strictly increasing between x_0 and x_2. □

Let Θ be a set (of parameters) and $g : \Theta \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be a measurable function. We denote by $Q_\tau[g(\theta, \cdot)|z]$ the quantile function associated with g, that is:

$$Q_\tau[g(\theta, \cdot)|z] \equiv \inf\{\alpha \in \mathbb{R} : \Pr([g(\theta, W) \leqslant \alpha]|Z = z) \geqslant \tau\}. \tag{48}$$

The following Lemma generalizes equation (2) to conditional quantiles.

Lemma 7.2. *Let* $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ *be non-decreasing and left-continuous in* $\mathcal{Z}$*. Then,*

$$Q_\tau[g(\theta, \cdot)|z] = g\left(\theta, Q_\tau[w|z]\right). \tag{49}$$

It is useful to illustrate the above result with an example. Let us define the function $g_{ab} : [1, 5] \to \mathbb{R}$ by:

$$g_{ab}(x) = \begin{cases} 7, & \text{if } x < a \\ b, & \text{if } x = a \\ 10, & \text{if } x > a \end{cases}$$

The function g_{ab} thus defined is non-decreasing if $b \in [7, 10]$ and it is left-continuous if $b = 7$.

Consider the r.v. X whose c.d.f. F and quantile function Q are shown in Figure 5 above. Let F_{ab} and Q_{ab} denote respectively the c.d.f. and quantile functions associated to $g_{ab}(Z)$. Figure 6 shows $Q_\tau[g_{ab}(w)|z]$ and $g_{ab}\left(Q_\tau[w|z]\right)$ for $a \in [1, 5]$ and $b \in [7, 10]$. The point of discontinuity is a function of a ($h(a) \in [0, 1]$).

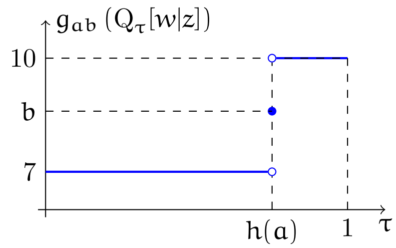


Figure 6a: $g_{ab}\left(Q_\tau[w|z]\right)$.

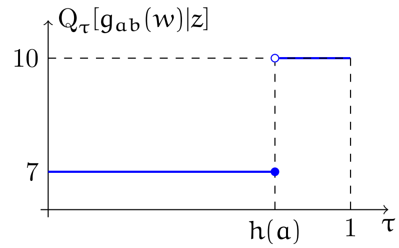


Figure 6b: $Q_\tau[g_{ab}(w)|z]$.

Proof of Lemma 7.2: For a contradiction, let us first assume that

$$Q_\tau[g(\theta, \cdot)|z] > g\left(\theta, Q_\tau[w|z]\right) \equiv \hat{\alpha}.$$

This means that $\hat{\alpha} \notin \{\alpha \in \mathbb{R} : \Pr\left(\{w : g(\theta, w) \leqslant \alpha\}|z\right) \geqslant \tau\}$, that is,

$$\Pr\left(\{w : g(\theta, w) \leqslant \hat{\alpha}\}|z\right) < \tau.$$

Since $\hat{\alpha} = g\left(\theta, Q_\tau[w|z]\right)$ and g is non-decreasing in w, $\{w : w \leqslant Q_\tau[w|z]\} \subset \{w : g(\theta, w) \leqslant \hat{\alpha}\}$. Thus, $\Pr\left(\{w : w \leqslant Q_\tau[w|z]\}|z\right) < \tau$, but this contradicts Lemma 7.1(iv).

Conversely, assume that

$$Q_\tau[g(\theta, \cdot)|z] < g\left(\theta, Q_\tau[w|z]\right).$$

This means that there exists $\tilde{\alpha} < g\left(\theta, Q_\tau[w|z]\right)$ such that

$$\Pr\left(\{w : g(\theta, w) \leqslant \tilde{\alpha}\}|z\right) \geqslant \tau.$$

Let $\tilde{w}$ be the supremum of the set $\{w : g(\theta, w) \leqslant \tilde{\alpha}\}$. Since g is non-decreasing and left-continuous, $g(\theta, \tilde{w}) \leqslant \tilde{\alpha}$. Moreover,

$$\Pr\left(\{w : w \leqslant \tilde{w}\}|z\right) = \Pr\left(\{w : g(\theta, w) \leqslant \tilde{\alpha}\}|z\right) \geqslant \tau.$$

Thus, $\tilde{w} \in \{\alpha \in \mathbb{R} : \Pr(\{w : w \leqslant \alpha\}|z) \geqslant \tau\}$, which implies that $\tilde{w} \geqslant Q_\tau[w|z]$. Thus, $\tilde{\alpha} \geqslant g(\theta, \tilde{w}) \geqslant g(\theta, Q_\tau[w|z]) > \tilde{\alpha}$, which is a contradiction. $\square$

The following Corollary to the above Lemma will be useful.

Corollary 7.3. *Let $T \in \mathbb{N} \cup \{\infty\}$, $h : \Theta \times \mathcal{Z}^T \times \mathcal{Z} \to \mathbb{R}$, $g : \Lambda \times \mathcal{Z}^T \times \mathcal{Z} \to \mathbb{R}$ be non-decreasing and left-continuous in $\mathcal{Z}$. Then,*

$$Q_\tau\left[h(\theta, z^T, Q_\tau[g(\lambda, z^T, z_{t+1})|z_t])|z_1\right] = Q_\tau\left[h\left(\theta, z^T, g(\lambda, z^T, Q_\tau[z_{t+1}|z_t])\right)|z_1\right].$$

Proof. Let X denote the random variable $Q_\tau[g(\lambda, z^t, z_{t+1})|z_t])$ and similarly, let Y denote $g(\lambda, z^t, Q_\tau[z_{t+1}|z_t])$. Then, by Lemma 7.2, $X = Y$. Therefore, $h(\theta, z^t, X) = h(\theta, z^t, Y)$ and the result follows. $\square$

The following result will be useful below.

Proposition 7.4. *Given the random variables X and Y, assume that there exists random variable Z and continuous and increasing functions h and g such that $X = h(Z)$ and $Y = g(Z)$. Then $Q_\tau[X+Y] = Q_\tau[X] + Q_\tau[Y]$.*

Proof. Let Z, h and g be as in the definition. Define $\tilde{h}(Z) \equiv h(Z) + g(Z)$. This function is clearly continuous and increasing. Therefore,

$$\begin{aligned} Q_\tau[X+Y] &= Q_\tau[h(Z)] = h(Q_\tau[Z]) = h(Q_\tau[Z]) + g(Q_\tau[Z]) \\ &= Q_\tau[h(Z)] + Q_\tau[g(Z)] = Q_\tau[X] + Q_\tau[Y]. \end{aligned}$$

by applying Lemma 7.2 twice. $\square$

7.2 Proofs of Section 3

Proof of Theorem 3.4: This is essentially the same proof of Theorem 3.9, presented in detail below. Thus, we omit it. $\square$

Proof of Proposition 3.5: Let L be a bound for V^π. Using repeated times the recursive property (13), we can write

$$\begin{aligned} V^\pi(x,z) &= u(x_1^\pi, x_2^\pi, z_1) + Q_\tau\Bigg[\beta u(x_2^\pi, x_3^\pi, z_2) + Q_\tau\Big[\beta^2 u(x_3^\pi, x_4^\pi, z_3) + \ldots \\ &\quad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^\pi, x_{n+2}^\pi, z_n) + \beta^{n+1} V^\pi(x_n^\pi, Z_n)\Big]\Big| Z_n = z_n\Big] \ldots \Big| Z_1 = z\Bigg] \\ &\leqslant u(x_1^\pi, x_2^\pi, z_1) + Q_\tau\Bigg[\beta u(x_2^\pi, x_3^\pi, z_2) + Q_\tau\Big[\beta^2 u(x_3^\pi, x_4^\pi, z_3) + \ldots \\ &\quad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^\pi, x_{n+2}^\pi, z_n) + \beta^{n+1} L\Big]\Big| Z_n = z_n\Big] \ldots \Big| Z_1 = z\Bigg] \\ &= V^n(x,z) + \beta^{n+1} L, \end{aligned}$$

where in the last line we have used the property of quantiles that $Q_\tau[X+\alpha] = \alpha + Q_\tau[X]$ for $\alpha \in \mathbb{R}$; see Lemma 7.2. Repeating the same argument with the lower bound $-L$, we can write:

$$V^n(x,z) - \beta^{n+1}L \leqslant V^\pi(x,z) \leqslant V^n(x,z) + \beta^{n+1}L.$$

This concludes the proof. $\square$

Proof of Proposition 3.7: Let $\Omega = \{1,2,3,4\}$ and $P(\{\omega\}) = 1/4$ for all $\omega \in \Omega$. Define $\Sigma_0 = \{\varnothing, \Omega\}$ and $\Sigma_1 = \{\varnothing, E_1, E_2, \Omega\}$, where $E_1 = \{1,2\}$ and $E_2 = \{3,4\}$. Let $X(\omega) = \omega$. Then for $\tau \in (0.5, 0.75)$,

$$Q_\tau[X|\Sigma_1]_\omega = \begin{cases} 2, & \text{if } \omega \in E_1 \\ 4, & \text{if } \omega \in E_2 \end{cases}$$

Therefore, $Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] = 4$ but $Q_\tau[X|\Sigma_0] = Q_\tau[X] = 3$, which establishes (19).

To see (20), consider $\Omega = [0,4]$, $\Sigma_0 = \{\varnothing, \Omega\}$ and let Σ_1 be generated by the partition $\{E_1, E_2\}$, where $E_1 = [1,2)$ and $E_2 = [2,4]$. Consider P as the uniform distribution on Ω. Let X and Y be two random variables with c.d.f. given respectively by $F_X(x) = \frac{1}{4}\left[x - \frac{1}{4}\sin(\pi x)\right]$ and $F_Y(x) = \frac{1}{4}\left[x + \frac{1}{4}\sin(\pi x)\right]$. The graphs of these two c.d.f.s are shown in Figure 7 below. Let $\tau \in (0.5, 0.75)$.

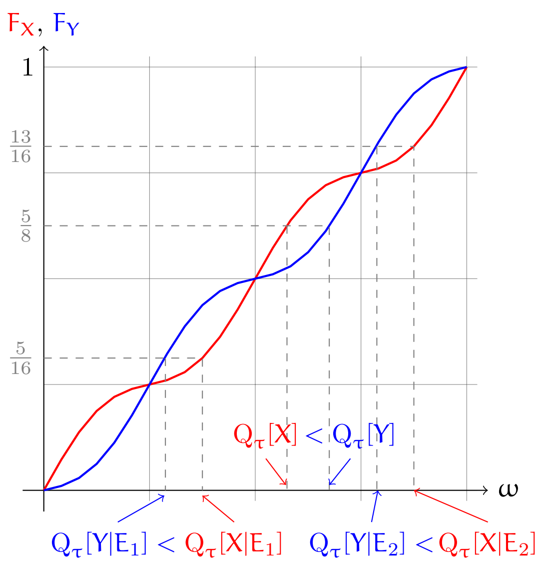


Figure 7: Graph of X and Y, with respective quantiles.

In the graph above, we plot the quantiles for $\tau = \frac{5}{8} \in (0.5, 0.75)$. We can easily see that $Q_\tau[X|\Sigma_1]_{(\omega)} \geqslant Q_\tau[Y|\Sigma_1]_{(\omega)}, \forall \omega \in \Omega$, but $Q_\tau[X] = Q_\tau[X|\Sigma_0] < Q_\tau[Y|\Sigma_0] = Q_\tau[Y]$, that is, (20) holds. $\square$

Proof of Theorem 3.8: Assume that plans π and π' are such that $\pi_{t'}(\cdot) = \pi'_{t'}(\cdot)$ for all $t' \leqslant t$ and $\pi' \succeq_{t+1,\Omega'_{t+1},x} \pi$ for all Ω'_{t+1}, x. From (9), this means that

$$V_{t+1}(\pi', x, z^{t+1}) \geqslant V_{t+1}(\pi, x, z^{t+1}), \forall (x, z^t) \in \mathcal{X} \times \mathcal{Z}^{t+1}. \tag{50}$$

Therefore,

$$\begin{aligned} V_t(\pi', x, z^t) &= u(x_t^{\pi'}, x_{t+1}^{\pi'}, z_t) + \beta Q_\tau \left[V_{t+1}(\pi', x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right] \\ &\geqslant u(x_t^{\pi'}, x_{t+1}^{\pi'}, z_t) + \beta Q_\tau \left[V_{t+1}(\pi, x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right] \\ &= u(x_t^{\pi}, x_{t+1}^{\pi}, z_t) + \beta Q_\tau \left[V_{t+1}(\pi, x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right] \\ &= V_t(\pi', x, z^t), \end{aligned}$$

where the first and last equalities come from the recursive equation (13), the first inequality comes from (50) and Lemma 7.1(vi), while the equality in the third line comes from the fact that the plans aggree on all times up to t, that is, $x_t^{\pi'} = x_t^{\pi}$ and $x_{t+1}^{\pi'} = \pi_t'(x_t^\pi, z^t) = \pi_t(x_t^\pi, z^t) = x_{t+1}^\pi$. This establishes the claim. □

Proof of Theorem 3.9: We organize the proof in a series of Lemmas.

Lemma 7.5. *If $v \in \mathcal{C}$, the map $(y, z) \mapsto Q_\tau[v(y, w)|z]$ is continuous.*

Proof. Consider a sequence $(y^n, z^n) \to (y^*, z^*)$. Since v and f are continuous, $v(y^n, w) \to v(y^*, w)$ and

$$m^n(\alpha) \equiv \Pr\left(\{w : v(y^n, w) \leqslant \alpha\} | z^n\right) \to \Pr\left(\{w : v(y^*, w) \leqslant \alpha\} | z^*\right) \equiv m^*(\alpha). \tag{51}$$

Let $\alpha^n \equiv \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geqslant \tau\} = Q_\tau[v(y^n, \cdot)|z^n]$ and $\alpha^* \equiv \inf\{\alpha \in \mathbb{R} : m^*(\alpha) \geqslant \tau\} = Q_\tau[v(y^*, \cdot)|z^*]$. We want to show that $\alpha^n \to \alpha^*$.

In general, $m^n(\cdot)$ and $m^*(\cdot)$ may fail to be continuous, but they are right-continuous and (weakly) increasing by Lemma 7.1. Moreover, m^* and m^n are strictly increasing in the range of v. More precisely, for each y, define $R(y) \equiv \{\alpha \in \mathbb{R} : \exists w \text{ such that } v(y, w) = \alpha\}$. We claim that if $\alpha < \alpha'$, $\alpha, \alpha' \in R(y)$, then $m^*(\alpha') > m^*(\alpha)$, and similarly for m^n.[22]

Indeed, assume that $\exists w, w'$ such that $v(y, w) = \alpha$ and $v(y, w') = \alpha'$. The set $P = \{\alpha w + (1-\alpha) w' : \alpha \in [0, 1]\}$ is contained in $\mathcal{Z}$ because this is convex. Thus, $\{v(y, p) : p \in P\}$ is connected, that is, a nonempty interval. We conclude that, since v is continuous, the set $\{w \in \mathcal{Z} : \alpha < v(y, w) < \alpha'\}$ is a nonempty and open interval. (This implies, in particular, that $R(y)$ is an interval.) Since $f(\cdot|z)$ is strictly positive in $\mathcal{Z}$, we conclude that

$$m^*(\alpha') - m^*(\alpha) \geqslant \Pr\left(\{w \in \mathcal{Z} : \alpha < v(y, w) < \alpha'\} | z\right) > 0,$$

which establishes the claim. By Lemma 7.1(iv), we have

$$m^n(\alpha^n) \geqslant \tau \text{ and } m^*(\alpha^*) \geqslant \tau. \tag{52}$$

We will show that $\alpha^n \to \alpha^*$ by first establishing $\liminf_n \alpha^n \geqslant \alpha^*$ and then $\alpha^* \geqslant \limsup_n \alpha^n$.

Suppose that $\underline{\alpha} \equiv \liminf_n \alpha^n < \alpha^*$. This means that there exists $\epsilon > 0$ and for each j, $n_j > j$ such that $\alpha^{n_j} < \underline{\alpha} + \epsilon < \alpha^*$. By the definition of α^*, $\underline{\alpha} < \alpha^*$ implies $m^*(\underline{\alpha}) < \tau$. However, by (52),

[22] Note that m^n and m^* are the corresponding c.d.f. functions for v. Thus, proving that those functions are strictly increasing in the range of v leads to continuity of the quantile with respect to τ, by (an adaptation of) Lemma 7.1(viii). But this is not what we need: we want continuity in (y, z). We prefer to offer here a direct and detailed argument, although long.

$m^{n_j}(\alpha^{n_j}) \geqslant \tau$, which implies $m^{n_j}(\underline{\alpha}) \geqslant \tau$ and $m^*(\underline{\alpha}) \geqslant \tau$, by (51). This contradiction establishes that $\liminf_n \alpha^n \geqslant \alpha^*$.

If $\bar{\alpha} \equiv \limsup_n \alpha^n > \alpha^*$, there exists $\epsilon > 0$ and for each j, $n_j > j$ such that

$$\bar{\alpha} + \epsilon > \alpha^{n_j} > \bar{\alpha} - \epsilon > \bar{\alpha} - 2\epsilon > \alpha^* + \epsilon. \tag{53}$$

Recall that $\alpha^n = \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geqslant \tau\}$. Therefore, $\alpha^{n_j} > \bar{\alpha} - \epsilon$ implies $m^{n_j}(\bar{\alpha} - \epsilon) < \tau$. Thus, $m^{n_j}(\alpha^* + \epsilon) < m^{n_j}(\bar{\alpha} - \epsilon) < \tau$. This implies that

$$m^*(\alpha^*) \leqslant m^*(\bar{\alpha} - 2\epsilon) \leqslant m^*(\bar{\alpha} - \epsilon) = \lim_n m^{n_j}(\bar{\alpha} - \epsilon) \leqslant \tau \leqslant m^*(\alpha^*).$$

Therefore, m^* is constant between α^* and $\bar{\alpha} - 2\epsilon$. This will be a contradiction if we show that $\alpha^*, \bar{\alpha} - 2\epsilon \in R(y^*)$.

Since $m^*(\alpha^*) = \Pr(\{w : v(y^*, w) \leqslant \alpha^*\}|z^*) \geqslant \tau > 0$, $\{w : v(y^*, w) \leqslant \alpha^*\} \neq \varnothing$ and there exists some $\alpha \in R(y^*) \cap (-\infty, \alpha^*]$. On the other hand, if $\{w : \bar{\alpha} - 2\epsilon \leqslant v(y^*, w) \leqslant \bar{\alpha} + 2\epsilon\} = \varnothing$, then for sufficiently high j, $\{w : \bar{\alpha} - \epsilon \leqslant v(y^{n_j}, w) \leqslant \bar{\alpha} + \epsilon\} = \varnothing$. In this case, $m^{n_j}(\bar{\alpha} - \epsilon) = m^{n_j}(\bar{\alpha} + \epsilon) \equiv \tau^{n_j}$. But this would imply either $\alpha^{n_j} \leqslant \bar{\alpha} - \epsilon$, if $\tau^{n_j} \geqslant \tau$ or $\alpha^{n_j} \geqslant \bar{\alpha} + \epsilon$, if $\tau^{n_j} < \tau$. In either case, we have a contradiction with $\alpha^{n_j} \in (\bar{\alpha} - \epsilon, \bar{\alpha} + \epsilon)$ as required in (53). This contradiction shows that there exists $\alpha' \in R(y^*) \cap [\bar{\alpha} - 2\epsilon, \bar{\alpha} + 2\epsilon]$. Since $\alpha, \alpha' \in R(y^*)$, we have $[\alpha^*, \bar{\alpha} - 2\epsilon] \subset [\alpha, \alpha'] \subset R(y^*)$. This concludes the proof. □

Lemma 7.6. *For each $v \in \mathcal{C}$ the supremum in (21) is attained and $\mathbb{M}^\tau(v) \in \mathcal{C}$. Moreover, the optimal correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by*

$$\Upsilon(x, z) \equiv \arg \max_{y \in \Gamma(x,z)} Q_\tau[u(x, y, z) + \beta v^\tau(y, w)|z] \tag{54}$$

is nonempty and upper semi-continuous.

Proof. Let

$$g(x, y, z, w) = u(x, y, z) + \beta v(y, w). \tag{55}$$

By Lemma 7.2, $Q_\tau[g(x, y, z, \cdot)|z] = u(x, y, z) + \beta Q_\tau[v(y, \cdot)|z]$. By Lemma 7.5, $Q_\tau[g(x, y, z, \cdot)|z]$ is continuous in (x, y, z). From Berge's Maximum Theorem, the maximum is attained, the value function $\mathbb{M}^\tau(v)$ is continuous and Υ is nonempty and upper semi-continuous. $\mathbb{M}^\tau(v)$ is bounded because u and v, hence g, are bounded. Therefore, $\mathbb{M}^\tau(v) \in \mathcal{C}$. □

We conclude the proof of Theorem 3.9 by showing that $\mathbb{M}^\tau$ satisfies Blackwell's sufficient conditions for a contraction.

Lemma 7.7. $\mathbb{M}^\tau$ *satisfies the following conditions:*

(a) For any $v, v' \in \mathcal{C}$, $v \leqslant v'$ implies $\mathbb{M}^\tau(v) \leqslant \mathbb{M}^\tau(v')$.

(b) For any $a \geqslant 0$ and $x \in X$, $\mathbb{M}(v + a)(x) \leqslant \mathbb{M}(v)(x) + \beta a$, with $\beta \in (0, 1)$.

Then, $\|\mathbb{M}(v) - \mathbb{M}(v')\| \leqslant \beta \|v - v'\|$, that is, $\mathbb{M}$ is a contraction with modulus β. Therefore, $\mathbb{M}^\tau$ has a unique fixed-point $v^\tau \in \mathcal{C}$.

Proof. To see (a), let $v, v' \in \mathcal{C}$, $v \leqslant v'$ and define g as in (55) and analogously for g', that is, $g'(x, y, z, w) = u(x, y, z) + \beta v'(y, w)$. It is clear that $g \leqslant g'$. Then, by Lemma 7.1(vi), $Q_\tau[g(\cdot)|z] \leqslant Q_\tau[g'(\cdot)|z]$, which implies (a).

To verify (b), we use the monotonicity property (Lemma 7.2):

$$Q_\tau[u(x, y, z) + \beta(v(x, z) + a)|z] \quad = \quad Q_\tau[u(x, y, z) + \beta v(x, z)|z] + \beta a.$$

Thus, $\mathbb{M}^\tau(v + a) = \mathbb{M}^\tau(v) + \beta a$, that is, (b) is satisfied with equality. □

Proof of Theorem 3.10: Let assumption 2 hold. It is convenient to introduce the following notation. Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ which are concave in its first argument. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly concave functions. If we show that $\mathbb{M}^\tau(\mathcal{C}') \subset \mathcal{C}''$, then the fixed-point of $\mathbb{M}^\tau$ will be strictly concave in x. (See, for instance, Stokey, Lucas, and Prescott (1989, Corollary 1, p. 52).)

Lemma 7.8. *Let assumption 2 hold.* $\mathbb{M}^\tau(\mathcal{C}') \subseteq \mathcal{C}''$. *Therefore,* $v^\tau \in \mathcal{C}''$. *Moreover, the optimal correspondence* $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ *defined by (54) is single-valued. Therefore, we can denote it by a function* $y^*(x, z)$.

Proof. Let $\alpha \in (0, 1)$, and consider $x_0, x_1 \in \mathcal{X}$, $x_0 \neq x_1$. For $i = 0, 1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, that is,

$$\mathbb{M}^\tau(v)(x_i, z) = u(x_i, y_i, z) + \beta Q_\tau[v(y_i, w)|z] = Q_\tau[g(x_i, y_i, z, w)|z].$$

Let $x_\alpha \equiv \alpha x_0 + (1 - \alpha)x_1$ and $y_\alpha \equiv \alpha y_0 + (1 - \alpha)y_1$. First, let us observe that

$$\begin{aligned} g(x_\alpha, y_\alpha, z, w) &= u(x_\alpha, y_\alpha, z) + \beta v(y_\alpha, w) \\ &> \alpha u(x_0, y_0, z) + (1 - \alpha)u(x_1, y_1, z) \\ &\quad + \beta v(y_\alpha, w) \\ &\geqslant \alpha u(x_0, y_0, z) + (1 - \alpha)u(x_1, y_1, z) \\ &\quad + \beta [\alpha v(y_0, w) + (1 - \alpha)v(y_1, w)] \\ &= \alpha g(x_0, y_0, z, w) + (1 - \alpha)g(x_1, y_1, z, w), \end{aligned}$$

where the first inequality comes from the strict concavity of u and the second, from the concavity of v. That is, g is strictly quasiconcave, which establishes that $\Upsilon(x, z)$ is single-valued. Therefore,

$$Q_\tau[g(x_\alpha, y_\alpha, z, w)|z] \quad > \quad Q_\tau[\alpha g(x_0, y_0, z, w) + (1 - \alpha)g(x_1, y_1, z, w)|z].$$

Note that the variables $X = g(x_0, y_0, z, w)$ and $Y = g(x_1, y_1, z, w)$ satisfy the assumption of Proposition 7.4, since v is nondecreasing in w (holding z fixed). Therefore,

$$\begin{aligned} Q_\tau[g(x_\alpha, y_\alpha, z, w)|z] &> \alpha Q_\tau[g(x_0, y_0, z, w)|z] + (1 - \alpha)Q_\tau[g(x_1, y_1, z, w)|z] \\ &= \alpha \mathbb{M}^\tau(v)(x_0, z) + (1 - \alpha)\mathbb{M}^\tau(v)(x_1, z). \end{aligned} \tag{56}$$

Therefore,

$$
\begin{aligned}
\mathbb{M}^{\tau}(v)(x_{\alpha}, z) &\geqslant Q_{\tau}[g(x_{\alpha}, y_{\alpha}, z, w)|z] \\
&> \alpha \mathbb{M}^{\tau}(v)(x_0, z) + (1-\alpha)\mathbb{M}^{\tau}(v)(x_1, z),
\end{aligned}
$$

This establishes strict concavity, concluding the proof. □

Lemma 7.9. *Let assumption 2 hold. If* $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leqslant z'$*, then* $Q_{\tau}[h(w)|z] \leqslant Q_{\tau}[h(w)|z']$.

Proof. From Assumption 2(ii), if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leqslant z'$:

$$\int_{\mathcal{Z}} h(\alpha)\left[-1_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}}\right] f(\alpha|z) d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha)\left[-1_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}}\right] f(\alpha|z') d\alpha.$$

Thus,

$$\int_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}} h(\alpha) f(\alpha|z) d\alpha \geqslant \int_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}} h(\alpha) f(\alpha|z') d\alpha. \tag{57}$$

If we define $H(w|z) = \Pr([h(W) \leqslant h(w)] | Z = z)$, then (57) can be written as:

$$H(w|z) \geqslant H(w|z').$$

Observe that $Q_{\tau}[h(w)|z] = \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\}$ and, whenever $z \leqslant z'$, $H(w|z') \leqslant H(w|z)$, for all w. Therefore, if $z \leqslant z'$, then

$$\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\} \supset \{\alpha \in \mathbb{R} : H(\alpha|z') \geqslant \tau\},$$

which implies that

$$Q_{\tau}[h(w)|z] = \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\} \leqslant \inf\{\alpha \in \mathbb{R} : H(\alpha|z') \geqslant \tau\} = Q_{\tau}[h(w)|z'],$$

as we wanted to show. □

Lemma 7.10. *Let assumption 2 hold. If* $v \in \mathcal{C}$ *is increasing in* z *then* $\mathbb{M}^{\tau}(v)$ *is strictly increasing in* z.

Proof. Let $z_1, z_2 \in \mathcal{Z}$, with $z_1 < z_2$. For $i = 1, 2$, let $y_i \in \Gamma(x, z_i)$ realize the maximum, that is,

$$\mathbb{M}^{\tau}(v)(x_i, z) = u(x, y_i, z_i) + \beta Q_{\tau}[v(y_i, w)|z_i].$$

Since u is strictly increasing in z, we have:

$$\mathbb{M}^{\tau}(v)(x, z_1) = u(x, y_1, z_1) + \beta Q_{\tau}[v(y_1, w)|z_1] < u(x, y_1, z_2) + \beta Q_{\tau}[v(y_1, w)|z_1].$$

From Lemma 7.9, we have $Q_{\tau}[v(y_1, w)|z_1] \leqslant Q_{\tau}[v(y_1, w)|z_2]$, which gives:

$$\mathbb{M}^{\tau}(v)(x, z_1) < u(x, y_1, z_2) + \beta Q_{\tau}[v(y_1, w)|z_2].$$

From Assumption 2, $\Gamma(x,z) \subseteq \Gamma(x,z')$, that is, $y_1 \in \Gamma(x,z_2)$. Optimality thus implies that:

$$u(x,y_1,z_2) + \beta Q_\tau[v(y_1,w)|z_2] \leqslant u(x,y_2,z_2) + \beta Q_\tau[v(y_2,w)|z_2] = \mathbb{M}^\tau(v)\,(x,z_2)\,.$$

Therefore, $\mathbb{M}^\tau(v)\,(x,z_1) < \mathbb{M}^\tau(v)\,(x,z_2)$, which shows strict increasingness in z. $\square$

We conclude the proof of Theorem 3.10 by showing differentiability of v, which follows from an easy adaptation of Benveniste and Scheinkman (1979)'s argument. For completeness and reader's convenience, we reproduce it here. Given (x,z), let $y^*(x,z) \in \Gamma(x,z)$ be unique maximum as established in Lemma 7.8. Thus, for all (x,z), we have:

$$v(x,z) = u(x,y^*(x,z),z) + \beta Q_\tau[v(y^*(x,z),w)|z].$$

Fix x_0 in the interior of X and define:

$$\bar{w}(x,z) = u(x,y^*(x_0,z),z) + \beta Q_\tau[v(y^*(x_0,z),w)|z].$$

From the optimality, for a neighborhood of x_0, we have $\bar{w}(x,z) \leqslant v(x,z)$, with equality at $x = x_0$, which implies $\bar{w}(x,z) - \bar{w}(x_0,z) \leqslant v(x,z) - v(x_0,z)$. Note that $\bar{w}$ is concave and differentiable in x because u is. Thus, any subgradient p of v at x_0 must satisfy

$$p \cdot (x - x_0) \geqslant v(x,z) - v(x_0,z) \geqslant \bar{w}(x,z) - \bar{w}(x_0,z).$$

Thus, p is also a subgradient of $\bar{w}$. But since $\bar{w}$ is differentiable, p is unique. Therefore, v is a concave function with a unique subgradient. Therefore, it is differentiable in x (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to x is the same as that of $\bar{w}$, that is,

$$\frac{\partial v^\tau}{\partial x_i}(x,z) = \frac{\partial \bar{w}}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x,y^*(x,z),z),$$

as we wanted to show. $\square$

Proof of Lemma 3.12: By Stokey, Lucas, and Prescott (1989, Theorem 7.6), Γ has a measurable selection. Therefore, the argument in Stokey, Lucas, and Prescott (1989, Lemma 9.1) establishes the result. $\square$

We need the following notation in the next proof. Let $T \in \mathbb{N} \cup \{\infty\}$ and $S : \mathcal{Z}^T \to \mathcal{Z}^{T-1}$ be the shift operator, that is, given $z = (z_1, z_2, ..., z_T) \in \mathcal{Z}^T$, $S(z) = (z_2, ..., z_T) \in \mathcal{Z}^{T-1}$. Abusing notation, let $S : \Pi \to \Pi$ also denote the shift operator for plans, that is, given $\pi \in \Pi$, $\pi^s = S(\pi) \in \Pi$ is defined as follows: for each given $z^\infty \in \mathcal{Z}^\infty$, $\pi^s_t(x, S(z^{t+1})) = \pi_{t+1}(x, z^{t+1})$. Let $S_t : \Pi \to \Pi$ be the composition of S with itself t times.

Proof of Lemma 3.13: Let $t \geqslant 2$ (otherwise there is nothing to prove). Since $\Pi_t(x,z) \subset \Pi_1(x,z) = \Pi(x,z)$ by definition, we have $v^*_t(x,z) \leqslant v_1(x,z)$. Suppose, for an absurd, that there exists $\pi \in \Pi(x,z)$ such that

$$V_1(\pi, x, z) > v^*_t(x,z). \tag{58}$$

Let $\tilde{\pi}$ and $(\tilde{x}, \tilde{z}^t)$ be such that $S_{t-1}(\tilde{\pi}) = \pi$, $x_t^{\tilde{\pi}}(\tilde{x}, \tilde{z}^t) = x$ and $\tilde{z}_t = z$. Then, $V_t(\tilde{z}, \tilde{x}, \tilde{z}^t) = V_1(\pi, x, z)$. Since $v_t^*(x, z) \geqslant V_t(\tilde{z}, \tilde{x}, \tilde{z}^t)$, this establishes a contradiction with (58). $\square$

Proof of Lemma 3.14: If v is bounded and satisfies (23), then it is the unique fixed-point of the contraction $\mathbb{M}^\tau$. Thus, the proof of Theorem 3.9 establishes, via the Maximum Theorem, the claims. $\square$

Proof of Theorem 3.15: Assume that v satisfies (23). It is sufficient to show that (i) $v(x,z) \geqslant V_1(\pi, x, z)$ for any $\pi \in \Pi(x,z)$ and $(x,z) \in \mathcal{X} \times \mathcal{Z}$; and (ii) $v(x,z) = V_1(\pi^\psi, x, z)$. Let $\pi \in \Pi(x,z)$. We have:

$$\begin{aligned}
v(x,z) &= \sup_{y \in \Gamma(x_1^\pi, z_1)} u(x_1^\pi, y, z_1) + \beta Q_\tau[v(y, z_2)|z_1] \\
&\geqslant u(x_1^\pi, x_2^\pi, z_1) + \beta Q_\tau [v(x_2^\pi, z_2)|z_1] \\
&= u(x_1^\pi, x_2^\pi, z_1) + \beta Q_\tau \left[\sup_{y \in \Gamma(x_2^\pi, z_2)} \left\{ u(x_2^\pi, y, z_2) + \beta Q_\tau \left[v(y, z_3) \middle| z_2 \right] \right\} \middle| z_1 \right] \\
&\geqslant u(x_1^\pi, x_2^\pi, z_1) + Q_\tau \left[\beta u(x_2^\pi, x_3^\pi, z_2) + Q_\tau \left[\beta^2 v(x_3^\pi, z_3) \middle| z_2 \right] \middle| z_1 \right],
\end{aligned}$$

where the two inequalities come from the definition of sup, and the equalities from (23) and Corollary 7.3. Repeating the same arguments, we obtain:

$$\begin{aligned}
v(x,z) \geqslant\; & u(x_1^\pi, x_2^\pi, z_1) + Q_\tau \Bigg[\beta u(x_2^\pi, x_3^\pi, z_2) + Q_\tau \Big[\beta^2 u(x_3^\pi, x_4^\pi, z_3) + \dots \\
& \dots + Q_\tau \left[\beta^n u(x_{n+1}^\pi, x_{n+2}^\pi, z_n) + \beta^{n+1} v(x_n^\pi, Z_n)] \middle| Z_n = z_n \right] \dots \Big| Z_1 = z \Bigg].
\end{aligned}$$

Repeating the arguments in the proof of Proposition 3.5, we can conclude that the limit of the right hand size when $n \to \infty$ is $V^\pi(x,z) = V_1(\pi, x, z)$. Thus, we have established that $v(x,z) \geqslant V_1(\pi, x, z)$. Since π was arbitrary, then $v(x,z) \geqslant v^*(x,z)$. On the other hand, for π^ψ the inequalities above hold with equality and we obtain $v(x,z) = v^*(x,z)$. $\square$

Proof of Theorem 3.16: Let $g(x,y,z,w) \equiv u(x,y,z) + \beta Q_\tau[v^\tau(y,w)|z]$ and $y^*(x,z)$ be an interior solution of the problem (23). Observe that v^τ is increasing in w, differentiable in its first variable and for $0 < x_i' - x_i < \epsilon$, for some small $\epsilon > 0$,

$$v^\tau(x_i', x_{-i}, z) - v^\tau(x_i, x_{-i}, z) = \int_x^{x'} \frac{\partial v^\tau}{\partial x_i}(\alpha, x_{-i}, z) d\alpha = \int_x^{x'} \frac{\partial u}{\partial x_i}(\alpha, x_{-i}, z) d\alpha$$

is increasing in z because $\frac{\partial u}{\partial x_i}$ is. Therefore, the assumptions of Proposition 3.17 are satisfied and we conclude that $\frac{\partial Q_\tau}{\partial x_i}[v^\tau(x,z)] = Q_\tau\left[\frac{\partial v^\tau}{\partial x_i}(x,z)\right]$. Since u is differentiable in y, so is g. Since $y^*(x,z)$ is interior, the following first order condition holds:

$$\frac{\partial g}{\partial y_i}(x, y^*(x,z), z, Q_\tau[w|z]) = \frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta Q_\tau[\frac{\partial v^\tau}{\partial x_i}(y^*(x,z), w)|z]) = 0.$$

Now we apply Theorem 3.10 and its expression: $\frac{\partial v^\tau}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x, y^*(x,z), z)$, to conclude that

$$\frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta Q_\tau \left[\frac{\partial u}{\partial x_i}(y^*(x,z), y^*(y^*(x,z), w)), w) \Big| z \right] = 0. \tag{59}$$

Now, we have just to put the notation of a sequence. For this, let $\pi = (x_t)$ denote an optimal path beginning at (x_0, z_0), (59) can be rewritten, substituting x for x_t^π, $y^*(x,z)$ for x_{t+1}^π, $y^*(y^*(x,z), w)$ for x_{t+2}^π, z for z_t and w for z_{t+1}, as:

$$\frac{\partial u}{\partial y_i}\left(x_t^\pi, x_{t+1}^\pi, z_t\right) + \beta Q_\tau \left[\frac{\partial u}{\partial x_i}\left(x_{t+1}^\pi, x_{t+2}^\pi, z_{t+1}\right) \Big| z_t \right] = 0. \tag{60}$$

which we wanted to establish. □

Proof of Proposition 3.17: Fix $x = (x_i, x_{-i})$, with the usual meaning and $\delta > 0$. Define $X = \tilde{h}(z) = h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ and $Y = \tilde{g}(z) = h(x_i, x_{-i}, z)$. Since h and $d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ are increasing in z by assumption, the random variables X and Y satisfy the assumptions of the previous proposition, which allows us to conclude that

$$\begin{aligned} Q_\tau[h(x_i + \delta, x_{-i}, z)] &= Q_\tau[X + Y] = Q_\tau[X] + Q_\tau[Y] \\ &= Q_\tau[h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)] + Q_\tau[h(x_i, x_{-i}, z)]. \end{aligned}$$

Therefore,

$$\frac{Q_\tau[h(x_i + \delta, x_{-i}, z)] - Q_\tau[h(x_i, x_{-i}, z)]}{\delta} = Q_\tau \left[\frac{h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)}{\delta} \right].$$

Taking the limit when $\delta \to 0$ on both sides above, we obtain:

$$\frac{\partial Q_\tau}{\partial x_i}[h(x,z)] = Q_\tau \left[\frac{\partial h}{\partial x_i}(x,z) \right],$$

as we wanted to show. □

7.3 Proofs of Section 5

Proof of Lemma 5.1: Assumption 1 (i)−(iii) and (v) are immediate. Since $\mathcal{Z}$ and $\mathcal{X}$ are bounded, and U and $z \mapsto z + p(z)$ are C^1, u is C^1 and bounded. Thus, Assumption 1 is satisfied. Similarly, Assumptions 2 are easily seen to be satisfied. It remains to verify the assumption of Theorem 3.16, namely that $\frac{\partial u}{\partial x_i}\left(x_t^\pi, x_{t+1}^\pi, z_t\right)$ is strictly increasing in z_t, which happens if and only if $\log \frac{\partial u}{\partial x_i}\left(x_t^\pi, x_{t+1}^\pi, z_t\right)$ is strictly increasing in z_t. Since

$$\log \frac{\partial u}{\partial x}(x,y,z) = -\gamma \log\left[z \cdot x + p(z) \cdot (x - y)\right] + \log\left(z + p(z)\right),$$

and $x_t^\pi = x_{t+1}^\pi = 1$, we need to verify only that $-\gamma \log\left[z \cdot x\right]' + \left[\log\left(z + p(z)\right)\right]' > 0$. This is equivalent to $\gamma < z\left[\log\left(z + p(z)\right)\right]'$, which is contained in Assumption 3(iv). □

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