Cooperation Dynamics in Repeated Games of Adverse Selection

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Abstract

We study a class of repeated games with Markovian private information and characterize optimal equilibria as players become arbitrarily patient. We show that seemingly non-cooperative actions may occur in equilibrium and serve as signals of changes in private information. Players forgive such actions, and use the information they convey to adjust their continuation play. However, to forgive is not to forget: players keep track of the number of aggressions and enter into a punishment phase if that number becomes suspiciously high. Our model explains features of long-run relationships that are only barely understood, such as equilibrium defaults, unilateral price cuts, collusive price leadership, graduated sanctions, and restitutions. We also explore a model in which interactions are frequent and show how increasing the persistence of the process of types reduces informational frictions.

Keywords: Repeated games, adverse selection, signaling, tacit collusion, price leadership, price cuts, equilibrium defaults, graduated sanctions.

1 Introduction

During trench warfare in the First World War (1914-1918), frontline soldiers often refrained from attacking the enemy, provided that their restraint was reciprocated by soldiers on the other side. Army commanders were aware of this tendency towards non-aggresion and would order raids to correct the "offensive spirit" of the troops (Ashworth 1980, Axelrod 1984). Enemy soldiers would generally not be able to discern if aggressions were caused by opportunistic behavior or by military orders, and would be reluctant to return to the non-aggressive behavior. However, cooperation generally restarted after some time had elapsed, and soldiers were successful at maintaining low levels of aggression for significant periods of time.

Cooperative relationships often exhibit this type of dynamics. For example, it has been long recognized that firms trying to avoid price competition cycle between high and low prices (Markham 1952, Bresnahan 1987, Scherer and Ross 1990), sovereign countries that default on

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their obligations are temporarily excluded from international capital markets but are eventually able to borrow again (Cole, Dow, and English 1995, Tomz 2012), managers and unions enter into labor conflicts which are generally followed by periods with high production and generous bonuses (Li and Matouschek 2013), and governments in a self-enforcing trade agreement often raise and lower their import tariffs, even though a high tariff may be detrimental to foreign partners (Bagwell and Staiger 2005).

In this paper, we shed light on these phenomena by studying a discrete-time infinitely-repeated game with (imperfectly) persistent private information. Two players make perfectly observable decisions at each round. Player 1 is privately informed about his own payoffs, which evolve according to a finite Markov chain. Importantly, no communication is allowed, which implies that player 2 can learn about player 1's types only by observing player 1's actions.

We show that optimal equilibrium dynamics may allow for apparent cooperation breaks (such as aggressions, price cuts, debt defaults) that serve as a signal of optimal continuation play. More generally, in our incomplete information model, equilibrium actions have an *informational content* that determines the most profitable course of play for the relationship. We show how optimal equilibria make use of endogenously generated information and explain behaviors that are difficult to square with existing models.

Our main theoretical result is the characterization of a class of Pareto-optimal equilibria as players become arbitrarily patient. This characterization reduces the problem of determining the informational content of the informed player's actions, as a function of history, to a dynamic programming equation defined on the set of total expected payoff functions. Our dynamic programing formulation is new to the literature and we provide several examples that prove its usefulness.

Section 2 illustrates our approach and results by studying a two-player two-action prisoners' dilemma with incomplete information. We assume that player 1 has private information about his cost of investment in a joint project. When his cost is low, the situation is a symmetric prisoners' dilemma game, and when his cost is high it is no longer socially desirable for the players to invest. Player 1's cost evolves with positive persistence. The problem is subtle because player 2 does not observe player 1's type, nor can player 1 communicate his cost. We find optimal equilibria in two steps.

First, we relax incentive constraints by allowing players to commit to strategies at the beginning of the game to maximize the sum of expected payoffs. We reformulate this problem as a dynamic programming problem having as state variable the belief about player 1's type conditional on public information. Two interesting optimal dynamics arise when incentives are ignored. Under *reactive-signaling dynamics*, the informed party keeps signaling his type while player 2 imitates the behavior of the informed player. Under *time-off dynamics*, a failure to invest by the high cost player 1 triggers a *waiting phase*. During the waiting phase, both players refrain from investing during a fixed number of periods. Once the waiting phase is over, player 2 and the low-cost player 1 resume investments and a waiting phase is restarted when player 1's cost becomes high again.

Second, we prove that optimal dynamics can actually be mimicked when incentives are taken into account. We build strategies in the repeated game that keep track of the informed player's actions and test whether they are sufficiently likely to come from the underlying optimal decision rule.[1] Optimal play may require that the informed agent plays apparently hostile or aggressive actions. The uninformed agent will forgive such behaviors and continue to play according to the optimal rule. However, *to forgive is not to forget*: the uninformed agent keeps track of the number of aggressions, and players enter a punishment phase if that number becomes suspiciously high.

This simple model shows that adverse selection and imperfect communication can restrict the set of equilibrium payoffs to a strict subset of feasible individually-rational payoffs even when players are arbitrarily patient (Radner, Myerson, and Maskin 1986, show a similar result for repeated games with moral hazard). In the optimal equilibrium, the informed party has to shirk in some rounds, and has to incur in *costly gestures* or *let time pass by* to persuade the uninformed player to resume investments. These dynamics imply substantive welfare costs, and optimal equilibrium payoffs are bounded away from first-best payoffs, even as the discount factor goes to 1. If the costs of incomplete information are large enough, the optimal equilibrium consists of repetitions of the static Nash equilibrium.

Sections 3 and 4 extend the analysis to general games of one-sided incomplete information. We formally establish the upper bound for equilibrium payoffs by studying an average-reward optimality equation (AROE) for a hidden-state Markov decision problem. The AROE is a Bellman equation tailored to study undiscounted dynamic models. The hidden variable in the Markov decision process is player 1's type, which together with controls determine a distribution over actions. Once actions are observed, they are used to update beliefs. Beliefs about player 1's type, given observed actions, are the state variable in the dynamic programming equation. At a more conceptual level, the AROE captures a basic trade-off between separating and pooling control rules. If player 1 pools given a public history, player 2 can better optimize his period payoffs. On the other hand, when different types of player 1 separate, continuation public beliefs are more precise and therefore the relationship gains from better information. We also show how, under ergodicity restrictions on the process of beliefs, strategies that forgive but do not forget can be designed to virtually attain the upper bound in the repeated game with low discounting.

Section 5 studies games with separating and monotonic dynamics. In these games, period payoffs have strictly increasing differences in actions and types, and player 1 has a set of actions which is sufficiently numerous. We show that player 1's actions are strictly increasing in his type and therefore he keeps signaling his current conditions. These results help explain a number of phenomena in long-run relationships that are only barely understood.

[1] The process of actions is not a Markov process, so it is hard to perform tests based on it. We sidestep this difficulty by testing observed actions conditional on *simulated* public beliefs. See Section 4 for details.

In Section 5.1, we use the results of Section 5 to characterize the optimal collusive scheme in a Bertrand game of differentiated products in which one of the firms has private information about his costs. Consistent with case studies (Marshall and Marx 2013), in our model *unilateral price changes* occur on the path of play. Our model can be interpreted as a model of *collusive price leadership* (Stigler 1947, Markham 1951, Scherer and Ross 1990), in which an uninformed firm follows the informed firm's price changes.[2] We show that the dynamics of price leadership may involve significant costs for leader and follower. When cost increases, the informed firm raises his price, and experiences a short-term loss until its price raise is matched by the follower. Likewise, the follower experiences a short-term loss when the leader lowers his prices after a cost decrease.[3] Our model therefore provides concrete answers to some unsettled issues in industrial organization and antitrust.[4]

We also apply our results to provide a rational to the commonly referred practice of *graduated sanctions* when collectively managing common-pool resources. As Ostrom's (1990) shows, in several successful long-run relationships, after a member breaks a norm, cheated partners mildly adjust their continuation actions. The use of severe punishments, like Nash reversion, is the exception rather than the norm. As Dixit (2009) explains, this evidence is difficult to reconcile with existent theoretical frameworks. Our model fills this gap. In Section 5.2, we specialize our model to a repeated collective action game in which player 1 has private information about the benefits of a project. On the path of play, the lower the action by the informed player, the lower the expectation the uninformed player has about the relationship conditions, and therefore the lower player 2's action. Player 1 can also make restitutions that positively affect player 2's current payoffs and his continuation beliefs and actions.

Section 6 refines our analysis by studying the game in Section 2 as interactions become more frequent. Following a tradition initiated by Abreu, Milgrom, and Pearce (1991), we observe that as interactions become more frequent not only the discount factor increases but also the process of hidden types becomes more persistent. We show that changing the persistence of the process of types has important effects on the dynamics of cooperation and equilibrium payoffs. In the limit, signaling becomes inexpensive compared to the benefits from more precise beliefs and, as a result, incomplete information has virtually no costs.

[2] Collusive price leadership is relevant in many industries. Allen (1976), for example, documents collusive price leadership in the market of steam turbine generators in the 1960s and 1970s. In Section 5.1 we discuss additional empirical evidence.

[3] These short-term losses are significant in many industries. Clark and Houde (2013) study gasoline prices in Quebec, and find that a small price premium (2 cents or more per liter) for a few hours can result in a significant reduction in a station's sales for the day (around 35% to 50%).

[4] Green and Porter (1984), Abreu, Pearce, and Stacchetti (1986), and Rotemberg and Saloner (1986) study collusive equilibria with high and low price cycles, in which price movements are simultaneous across firms. Thus, there are no unilateral price changes or price leaders. Rotemberg and Saloner (1990) study collusive price leadership in a repeated Bertrand game, imposing exogenous constraints on strategies and on the timing of the game, and find that the leader always benefits more from price leadership than the follower. All these models have Pareto efficient equilibria if players are sufficiently patient. We study the optimal equilibrium without imposing exogenous restrictions, and show that price leadership can arise as an equilibrium outcome. This equilibrium may involve significant profit losses for both the leader and the follower, and as a result, may be inefficient even as the discount factor goes to 1.

When players can exchange cheap-talk messages right before choosing actions, Escobar and Toikka (2013) and Hörner, Takahashi, and Vieille (2015) show that the folk theorem obtains. With communication, actions have no signaling content and the dynamics of cooperation are similar to those of games with complete information and changing types if players are sufficiently patient (Rotemberg and Saloner 1986, Dutta 1995). In those models, actions can perfectly respond to current conditions and there is no room (on the path of play) to observe hostile behaviors (as we do in reality). Our assumption of no communication is just a simplifying one, and acknowledges the fact –articulated by Marschak and Radner (1972) among others– that oftentimes parties encounter nontrivial communication costs.[5]

Our results connect to the literature on repeated games with Markovian hidden types. Escobar and Toikka (2013), Renault, Solan, and Vieille (2013), and Hörner, Takahashi, and Vieille (2015) characterize optimal equilibria in games with communication. As explained above, dynamics in these models are very different from the ones in this paper. Athey and Bagwell (2001) and Athey and Bagwell (2008) characterize optimal equilibria in Bertrand games without communication, but their analysis exploits the special structure of their inelastic demand model. Hörner, Rosenberg, Solan, and Vieille (2010) study equilibrium values in the zero-sum case.[6] Our contribution is to characterize optimal equilibrium in a fairly general class of games with hidden types and no communication.

Other papers have also focused on defaults and cooperation cycles. Liu (2011) and Liu and Skrzypacz (2014) study games between a long-run player and a sequence of short-run players. The long-run player can be opportunistic or behavioral, and this is defined once and for all at the beginning of the game. Short-run players cannot freely access to the whole history of actions. This generates cycles of cooperation in which the long-run player builds and exploits his reputation.[7] Acemoglu and Wolitzky (2014) study a reputation model in which players have limited and noisy observations. In all these models, memory restrictions play a key role determining cycles. The force in our model is unrelated to memory limits.

We finally observe that in games with imperfect monitoring, players can also cycle between cooperative and uncooperative actions (Green and Porter 1984, Abreu, Pearce, and Stacchetti 1986, Abreu, Pearce, and Stacchetti 1990, Abreu, Milgrom, and Pearce 1991), but equilibrium dynamics differ significantly from the ones presented in our paper.

Green and Porter (1984) and Abreu, Pearce, and Stacchetti (1986) study repeated games with quantity competition, and characterize equilibria with high and low price regimes. Transitions between regimes depend on the realization of an exogenous random factor affecting demand. We show that in the case of adverse selection, regime changes depend on players'

[5]Price discussions between competitors are generally illegal. Ashworth (1980) documents the communication problems faced by enemy troops trying to avoid confrontation during World War I. When discussing limited war, Schelling (1960) explains that "an agreement on limits is difficult to reach ... because communication becomes difficult between adversaries in a time of war."

[6]Other papers studying repeated games with Markovian types include Gale and Rosenthal (1994), Cole, Dow, and English (1995), and Phelan (2006). These papers focus on specific equilibria that are typically bounded away from the Pareto-frontier.

[7]In those models, defaults are strategic while in our model defaults are mainly non-strategic.

actions. For example, low-price regimes (price wars) may be triggered by price cuts, and returning to high-price regimes may require unilateral price rises.

Abreu, Milgrom, and Pearce (1991) studies a prisoners' dilemma with imperfect monitoring and shows that, under certain conditions, cooperation can be broken and never resumed in the optimal equilibrium. There is therefore room for renegotiating punishments. In our model, in contrast, virtually no value is burnt (optimal equilibria sustains an informationally-constrained welfare optimum) and there is little room for renegotiation.

2 An Example

Two players, $i = 1, 2$, interact repeatedly in a public-good investment game. Every period, players decide whether to invest (I) or not to invest (N). The investment may represent an advertising expenditure in a joint-advertising campaign, an investment in R&D in a research joint venture, or costly effort in team of co-workers.

Stage payoffs are equal to investment revenues minus cost. If both players invest, each player obtains a revenue of a. If only one player invests, each player obtains a revenue of b. If no player invests, both players obtain zero revenues. Let $0 < b < a$. Player 1's investment cost in period t is $\theta^t \in \{l, h\}$, where $l < h$, and Player 2's investment cost is l every period. Table 1 summarizes game payoffs.

$\theta^t = l$:

	I	N
I	$a-l\,,\,a-l$	$b-l\,,\,b$
N	$b\,,\,b-l$	$0\,,\,0$

$\theta^t = h$:

	I	N
I	$a-h\,,\,a-l$	$b-h\,,\,b$
N	$b\,,\,b-l$	$0\,,\,0$

Table 1: Player 1's cost is privately known. Joint investment is socially desirable only when $\theta^t = l$.

The outcome (I, I) (resp. (N, N)) is socially desirable if and only if $\theta = l$ (resp. $\theta = h$). From a strategic perspective, for $\theta = l$ the game is a prisoners' dilemma, whereas for $\theta = h$ the only Nash equilibrium coincides with the socially desirable outcome (N, N). We capture these restrictions by assuming that $2(a - l) > 0$, $2a - l - h < 0$, $2b - l < 0$, and $a - l < b$. Together, these assumptions imply that $0 < b < \frac{l}{2} < l < a < \frac{l+h}{2} < h$ and $a - l < b$.

The cost parameter θ^t is realized at the beginning of period t and is privately known by player 1. Once player 1 privately observes his type, θ^t, players simultaneously choose actions. Monitoring is perfect, that is, actions are publicly observed. Player 1's type evolves according to a Markov process with transition probabilities given by

$$\begin{aligned} \mathrm{P}[\theta^t = l \mid \theta^{t-1} = l] &= \lambda \geq \frac{1}{2} \\ \mathrm{P}[\theta^t = h \mid \theta^{t-1} = h] &= \mu. \end{aligned}$$

where $\lambda + \mu \geq 1$ or, equivalently, the process of types has positive persistence. For simplicity, we assume that the initial type is drawn according to $\mathrm{P}[\theta^1 = l] = \lambda$. Players have a common discount factor $\delta < 1$ and maximize the discounted sum of period payoffs. This is a repeated game with Markovian incomplete information. Whereas player 1 knows that whole history of transpired types and play, player 2 can condition his behavior only on the history of actions.

It is worth pointing out two benchmarks that are relatively easy to solve. With complete information, the type of player 1, θ^t, is publicly observed at the beginning of round t. If δ is large enough, we can construct a trigger-strategy equilibrium in which play is efficient and both players invest in t if and only if $\theta^t = l$ (Rotemberg and Saloner 1986, Dutta 1995). Another interesting benchmark is the case of incomplete information and communication, in which player 1 is privately informed about θ^t but can send a cheap-talk message to player 2 before actions are decided. If δ is sufficiently big, one can construct an efficient equilibrium in which player 1 truthfully reveals his type and both players invest only when $\theta^t = l$ (Escobar and Toikka 2013). In this paper, we are interested in characterizing the optimal equilibrium with incomplete information and no communication.

Before describing equilibrium behavior, let us characterize optimal dynamics ignoring incentive constraints. Observe that even when incentives are ignored and player 1 can make use of all available information (the history of play and his privately-held information), player 2 can only condition his behavior on public information (the history of play). To study this problem, we introduce controls. A control for player 1 is a pair $\sigma_1 = (\sigma_1(l), \sigma_1(h)) \in \{I, N\}^2$, whereas a control for 2 is simply $\sigma_2 \in \{I, N\}$. A control $\sigma = (\sigma_1, \sigma_2)$ determines total period total payoffs, given beliefs. If the control is INI and the public information determines $p^t = \mathbb{P}[\theta^t = l]$, then expected period payoffs are

$$p^t\big((1-\delta)2(a-l)\big) + (1-p^t)\big((1-\delta)(2b-l)\big).$$

But a control also determines player 2's continuation beliefs, given a current distribution on 1's types. For example, suppose the belief at time t is $\mathrm{P}(\theta^t = l) = p^t$. Given this belief, if player 1's control has $\sigma_1^t(l) = I$ and $\sigma_1^t(h) = N$ (that is, player 1 invests if and only if her type is l); then period $t+1$'s probabilities depend on the action of player 1 at period t:

$$\begin{aligned}
\mathrm{P}_\sigma[\theta^{t+1} = l \mid a_1^t = I] &= \lambda \\
\mathrm{P}_\sigma[\theta^{t+1} = l \mid a_1^t = N] &= 1 - \mu.
\end{aligned}$$

If player 1's control has $\sigma_1^t(l) = \sigma_1^t(h) = N$ instead (that is, player 1 does not invest for any type realization); then period $t+1$ probabilities are given by:

$$\mathrm{P}_\sigma[\theta^{t+1} = l \mid a_1^t = N] \;=\; p^t\,\lambda + (1-p^t)\,(1-\mu).$$

Thus, for player 2, the probability that player 1's type is l in period $t+1$ depends on the control rule in place in period t and the observation made in t.

We consider the belief held by player 2 at t about 1's type, $p^t = \mathrm{P}(\theta^t = l)$, as a state variable. A $\sigma = (\sigma_1(l), \sigma_1(h), \sigma_2)$ will be characterized by a triple XYZ. Let $w(p)$ be the value for the problem of maximizing total discounted expected payoffs given beliefs p over all possible strategies. The above discussion leads to the following dynamic programming characterization for the value $w(p)$:

$$w(p) = \max \Big\{ w_{XYZ}(p) \mid X, Y, Z \in \{I, N\} \Big\}, \tag{2.1}$$

where

$$\begin{aligned}
w_{III}(p) &= p\,(1-\delta)\,2(a-l) + (1-p)\,(1-\delta)\,(2a-l-h) + \delta\, w(p\,\lambda + (1-p)(1-\mu)),\\
w_{IIN}(p) &= p\,(1-\delta)\,(2b-l) + (1-p)\,(1-\delta)\,(2b-h) + \delta\, w(p\,\lambda + (1-p)(1-\mu)),\\
w_{NNN}(p) &= (1-\delta)\,0 + \delta\, w(p\,\lambda + (1-p)(1-\mu)),\\
w_{NNI}(p) &= (1-\delta)\,(2b-l) + \delta\, w(p\,\lambda + (1-p)(1-\mu)),\\
w_{INI}(p) &= p\,((1-\delta)\,2(a-l) + \delta\, w(\lambda)) + (1-p)\,((1-\delta)\,(2b-l) + \delta\, w(1-\mu)),\\
w_{INN}(p) &= p\,((1-\delta)\,(2b-l) + \delta\, w(\lambda)) + (1-p)\,((1-\delta)\,0 + \delta\, w(1-\mu)),\\
w_{NII}(p) &= p\,((1-\delta)\,(2b-l) + \delta\, w(\lambda)) + (1-p)\,((1-\delta)\,(2a-l-h) + \delta\, w(1-\mu)),\\
w_{NIN}(p) &= p\,((1-\delta)\,0 + \delta\, w(\lambda)) + (1-p)\,((1-\delta)\,(2b-h) + \delta\, w(1-\mu)).
\end{aligned}$$

The optimal choice of control will trade-off current payoffs and the distribution over continuation beliefs. It is straightforward to see that controls IIN, NNI, NII, and NIN are never optimal. For example, under control NNI player 1 does not invest but player 2 does. This provides less period payoffs than the pooling control NNN. Since both controls determine the same distribution over continuation beliefs, control NNI cannot be optimal for any belief p.

The following lemma summarizes some important properties of $w(p)$. All omitted proofs are in the Appendix.

Lemma 1. *$w(p)$ is nondecreasing, continuous, and convex.*

To understand the convexity property, fix beliefs $p = \mathrm{P}(\theta^t = l)$ yielding value $w(p)$. Suppose now that we are offered more detailed information about this probability: we are told that $\mathrm{P}(\theta^t = l) = q$ with probability π and $\mathrm{P}(\theta^t = l) = q'$ with probability $1 - \pi$, such that $p = \pi q + (1-\pi)q'$. Now, value is $w(q)$ with probability π and $w(q')$ with probability $1 - \pi$. Convexity implies that we always prefer to have more information:

$$w(p) = w(\pi q + (1-\pi)q') \le \pi\, w(q) + (1-\pi)\, w(q').$$

Intuitively, when information is revealed, the optimal control can be adjusted to yield better outcomes.

The solution to (2.1) can result in pooling dynamics, in which players play a fixed action profile in all rounds. Assuming $a - b < h/2$, the pooling control III is never optimal as the incremental social benefit of 1's investment is low compared to his high cost h.[8]. When solving (2.1), it is enough to focus on controls NNN, INI, and INN.

Lemma 2 shows that optimal dynamics take very simple forms. The second-best rule generates *reactive-signaling* dynamics if player 1 invests when his cost is low and does not invest when his cost is high, and player 2 imitates the action of player 1 in the previous period. The second-best rule generates *time-off* dynamics if player 1 invests only if he is in good standing and his cost is low, and player 2 invests if and only if player 1 is in good standing. Player 1 is in good standing if he invested in the previous period, or if he did not invest in the previous period, but was in good standing $\hat{\tau} + 1$ periods before, where $\hat{\tau}$ is a natural number (possibly equal to 0).

Lemma 2. *If $a - b < h/2$ and $\lambda > \frac{l-2b}{2(a-b)-l}$, the second-best rule generates reactive-signaling or time-off dynamics.*

The restriction to $\lambda > \frac{l-2b}{2(a-b)-l}$ ensures that the control INI is optimal at belief $p = \lambda$. The Lemma shows that if player 1 does not invest at belief λ, then player 1 will "signal" a change of type by investing to prompt player 2 to invest too, or, alternatively, player 2 will wait for $\hat{\tau}$ rounds to become optimistic about player 1's cost and resume investments. This result rules out dynamics in which signaling can occur only after an exogenous number of rounds.

The choice between reactive signaling and time-off dynamics depends on the comparison between the *signaling cost* and the *opportunity cost* of missed cooperation. The signaling cost is $l - 2b$, which is the welfare loss suffered when only one player invests. Under reactive signaling dynamics, players incur in a signaling cost every time player 1's type changes. Under time-off dynamics, players incur in a signaling cost when player 1's type goes from l to h, and when a waiting phase ends, if the type of player 1 is h. Observe that under time-off, players may not incur in the signaling cost when a waiting phase ends, because the type of player 1 may be l when the waiting phase ends. The opportunity cost of missed cooperation is $2(a-l)$, and is the gain in welfare that would have accrued if both players had invested and cost was low. The cost of missed cooperation is incurred during a waiting phase in a time off rule. The total expected opportunity cost of missed cooperation depends on the optimal length of the waiting phase.

Let $\beta = \frac{l-2b}{2(a-l)}$ measure the signaling cost relative to the opportunity cost of missed cooperation. The following lemma shows how optimal dynamics depend on the parameters of the model as the discount factor goes to 1.

Lemma 3. *Assume that $a - b < h/2$ and $1 < (\lambda + \mu)(1 - \frac{\lambda}{2})$. There exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, there exists $\beta_0 \in]0, \frac{\lambda}{2(1-\lambda)}[$ such that*

[8]Another reason III is not optimal is that it does not improve information in the continuation nodes

i. For $\beta < \beta_0$, the optimal rule generates time-off dynamics;

ii. For $\beta \in]\beta_0, \frac{\lambda}{2(1-\lambda)}[$, the optimal rule generates reactive signaling dynamics;

iii. For $\beta > \frac{\lambda}{2(1-\lambda)}$, the optimal rule generates a path in which no player ever invests.

This lemma fully characterizes optimal dynamics under the added assumption that the process of types is persistent enough (so that $1 < (\lambda+\mu)(1-\frac{\lambda}{2})$).[9] When the signaling cost is small, time-off is optimal as even a short waiting period ensures that the benefits of mutual cooperation are realized. As β increases, the waiting period in time-off dynamics increases, and it is better to capture the benefits of cooperation by allowing player 1 to signal his changes in types. When the signaling costs are too high, there is no room for cooperation and players payoffs are 0.[10]

So far, we have ignored incentive problems. When thinking about the incentives player 1 would face to play according to optimal dynamics, one could be tempted to argue that those dynamics effectively "punish" a defection by player 1. For example, under reactive signaling, if player 1 does not invest when his cost is low, player 2 reacts by not investing in the next period, and this will provide incentives to player 1 to invest if and only if his cost is low. However, this type of argument can work only for some parameter values as reactive signaling and time-off dynamics arise to optimize the discounted sum of payoffs subject to informational constraint, but do not take into account incentives. It is therefore not obvious whether optimal dynamics payoffs can be attained when incentives are taken into account.

We now show it is always possible to find equilibrium strategies such that equilibrium play is arbitrarily close to second-best optimal dynamics. For concreteness, we take parameters such that, according to Lemma 3, optimal dynamics are reactive signaling. Under reactive signaling, the process of beliefs $(p^t)_{t\geq 1}$, with $p^t = \mathrm{P}[\theta^t = l \mid h^{t-1}]$ and h^{t-1} the public history up to and including round $t-1$, is Markovian, with transitions that can be drawn as shown in Figure 1.

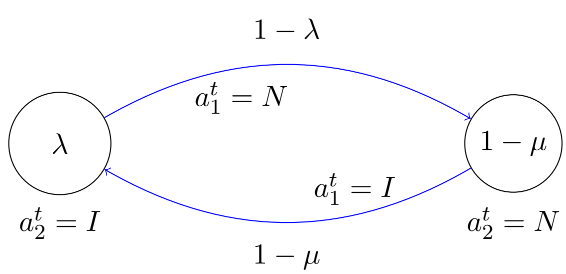


Figure 1: Dynamics of beliefs $(p^t)_{t\geq 1}$ when players use an optimal rule resulting in reactive signaling dynamics. The support of $(p^t)_{t\geq 1}$ is the set $\{\lambda, 1-\mu\}$.

[9] In the Appendix we dispense with this restriction and provide a complete characterization of optimal dynamics

[10] Observe that the bounds in the Lemma improve upon the restrictions in Lemma 2 to have some cooperation.

Let $v_i \in \mathbb{R}$ be the limit average payoff accruing to player i under the reactive signaling rule and assume $v_i > 0$. This means that both players get more payoffs from the optimal dynamics than from the static Nash equilibrium resulting in payoffs $(0, 0)$.

Ensuring appropriate behavior by player 2 is simple as any deviation by 2 is observable and can be immediately punished by reverting to the static Nash equilibrium. Incentives for player 1 are more subtle because player 2 cannot observe player 1's type, nor can he tell whether a failure to invest by player 1 is acceptable (because player 1's cost is high) or not. However, as play transpires, player 2 can keep checking whether player 1's behavior seems likely to have been generated from the reactive signaling rule. By simulating a process of beliefs from player 1's actions, the uninformed player 2 can check whether the proportions of investment and no-investment actions seem credible, conditional on a simulated belief. For example, out of all the visits to state λ, player 2 can check whether player 1 has played I in a proportion close to λ. A failure to do so would be observable and easily punished by Nash reversion.

The strategies discussed above continuously check whether players' actions seem credible. They are similar to strategies used in repeated games with imperfect monitoring (Radner 1981) and in dynamic mechanism design (Jackson and Sonnenschein 2007, Escobar and Toikka 2013).[11] In our construction of strategies, while player 2 can tolerate some failures (i.e., periods in which player 2 invested but player 1 did not), he keeps track of the number of offenses, and players enter a punishment phase if that number becomes suspiciously high.[12]

Our analysis has three main implications. First, as we explained above, informational constraints are key to determine optimal equilibrium dynamics. While incentive problems disappear as players become more patient, optimal equilibrium are bounded away from first-best payoffs. Indeed, with complete information (or with incomplete information and communication), players can attain average total payoffs equal to $2(a-l)\frac{1-\mu}{2-\lambda-\mu}$. Assuming the conditions under which reactive-signaling is optimal in Lemma 2, under incomplete information total payoffs are $\big(2\lambda(a-l)-(l-2b)(1-\lambda)\big)\frac{1-\mu}{2-\lambda-\mu}$. Moreover, when the signaling costs are too high, the only equilibrium of the game is the repetition of the static Nash equilibrium even when the discount factor is arbitrarily close to 1.[13] While communication obviously expand the set of equilibria, we seem to be the first ones fully characterizing the gains from communication in a repeated game model.[14]

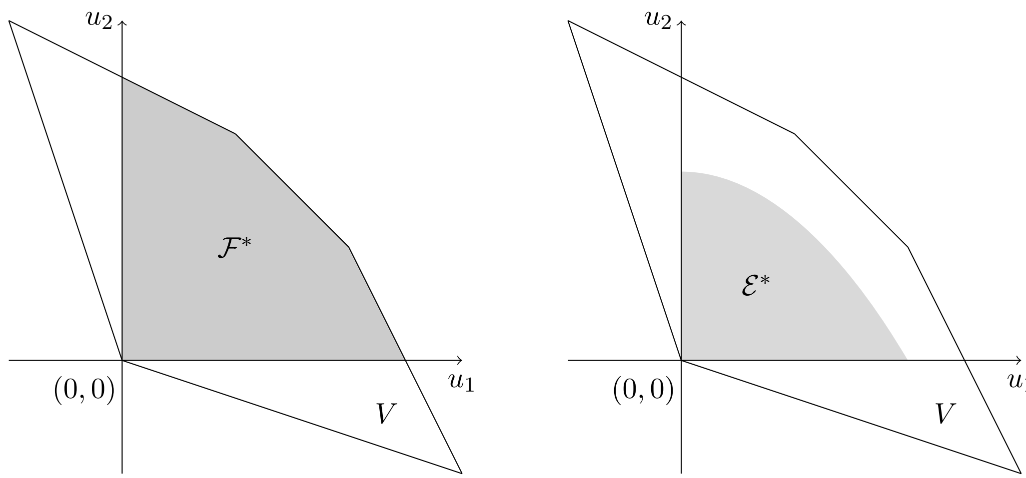
Second, cooperation dynamics differ from those found in previous papers. For example,

[11]As in all these papers, our strategies are derived from a test based on necessary conditions for "appropriate behavior", and we then show that these conditions are actually sufficient to align incentives.

[12]In this example, punishments simply consist in Nash reversion. In the general model of Section 3, punishments are more complex in order to guarantee that adhering to these punishments is incentive compatible for both players.

[13]As Hörner, Takahashi, and Vieille (2015) show, the set of equilibrium payoffs in the game with communication depends on the transitions only through the invariant distribution (Corollary 3). In contrast, in our model without communication transitions do matter to determine the equilibrium set.

[14]Awaya and Krishna (2014) study a repeated Bertrand game with imperfect private monitoring and show conditions under which the set of equilibrium payoffs without communication is strictly contained in the set of equilibrium payoffs with communication. They obtain a lower bound for the gap.



(a) $\mathcal{F}^*$ is the set of limit equilibrium payoffs in the game with complete information or incomplete information and communication. It contains all feasible payoffs above the minmax vector $(0, 0)$.

(b) $\mathcal{E}^*$ is the set of limit equilibrium payoffs in the game with incomplete information and no communication. It is strictly contained in $\mathcal{F}^*$. When signaling is too costly, as detailed in Lemma 2, $\mathcal{E}^* = \{(0, 0)\}$.

Figure 2: The equilibrium sets for games with and without communication.

when monitoring is imperfect, punishments are triggered on the equilibrium path and cooperation may be resumed exogenously (following a randomization device as in Abreu, Pearce, and Stacchetti 1990) or not resumed at all (as in Abreu, Milgrom, and Pearce 1991). In our model, actions have signaling content and cooperation is always resumed, either by taking a costly action (as in reactive signaling) or after a cooling-off period has elapsed (as in time-off dynamics). Since continuation payoffs are always close to optimal, virtually no value is burnt on the path of play and, in contrast to models with imperfect monitoring, there is little room for renegotiation.

Third, under reactive signaling, the on-path behavior of player 2 is identical to a tit-for-tat strategy. While tit-for-tat is an intuitive strategy and has received attention in the literature (Axelrod 1984, Kalai, Samet, and Stanford 1988, Kreps, Milgrom, Roberts, and Wilson 1982), no equilibrium framework exists under which it emerges as the optimal outcome. Our results fill this gap by showing how informational frictions make tit-for-tat a desirable strategy.

3 Model

We consider an infinitely repeated game played by 2 players. At each $t \geq 1$, player 1 is privately informed about his type $\theta^t \in \Theta$. Players simultaneously make decisions $a_i^t \in A_i$. Let $A = A_1 \times A_2$. We assume that A_1, A_2, and Θ are finite sets. Within each round t, play transpires as follows:

t.0 A randomization device χ^t is publicly realized t.1 Player 1 is privately informed about $\theta^t \in \Theta$

t.2 Players choose actions $a_i^t \in A_i$ simultaneously

t.3 Players observe the action profile chosen $a^t \in A$

We assume players know their payoffs. The period payoff function for player 1 is $u_1(a, \theta)$, whereas player 2's payoff is $u_2(a)$. We will sometimes abuse notation and write $u_i(a, \theta)$, even when player 2's payoff does not depend on θ. Players rank flows of payoffs according to $(1-\delta)\sum_{t\geq 1}\delta^{t-1}u_i(a^t, \theta^t)$, where $\delta < 1$ is the common discount factor.

The realizations of the randomization device are independent across time and distributed according to a uniform in $[0, 1]$. The initial type of player 1, θ^1, is drawn from a distribution $p^1 \in \Delta(\Theta)$. The process $(\theta^t)_{t\geq 1}$ evolves according to a Markov chain with transition matrix P (Norris 1997). Player 1's private types, $(\theta^t)_{t\geq 1}$, evolve according to a Markov chain (p^1, P), where $p^1 \in \Delta(\Theta)$ and P is a transition matrix on Θ. We assume that the process of types has full support. This means that for all $\theta, \theta' \in \Theta$, $P(\theta' \mid \theta) > 0$. Let $\pi \in \Delta(\Theta)$ be the stationary distribution for P.

A strategy for player 1 is a sequence of functions $s_1 = (s_1^t)_{t\geq 1}$ with $s_1^t \colon \Theta^t \times A^{t-1} \times [0, 1]^t \to A_1$, whereas a strategy for the uninformed player 2 is $s_2 = (s_2^t)_{t\geq 1}$ with $s_2^t \colon A^{t-1} \times [0, 1]^t \to A_2$. A strategy profile $s^* = (s_1^*, s_2^*)$ is a perfect Bayesian equilibrium if there exists a system of beliefs constructed from Bayes rule (when possible) such that s_i^* is sequentially rational (Fudenberg and Tirole 1991). The set of perfect Bayesian equilibrium payoffs will be denoted $\mathcal{E}(\delta, p^1) \subseteq \mathbb{R}^2$.

A *decision rule* is a sequence $f = (f^t)_{t\geq 1}$ with $f^t = (f_1^t, f_2^t)$ and $f_1^t \colon A^{t-1} \times \Theta^t \times \times [0, 1]^t \to \Delta(A_1)$ and $f_2^t \colon A^{t-1} \times [0, 1]^t \to \Delta(A_2)$. A decision rule determines a possibly mixed action for each player i as a function of the publicly observed history of action profiles and randomizations, and his own private history of realized types. Any decision rule f induces a probability distribution over histories. We can therefore define the vector of expected payoffs given a decision rule f as

$$v^\delta(f) = (1-\delta)\mathbb{E}_f[\sum_{t\geq 1}\delta^{t-1}u(a^t, \theta^t)] \in \mathbb{R}^2.$$

Let $V(\delta, \lambda) = \{v = v^\delta(f) \in \mathbb{R}^2 \text{ for some decision rule } f\}$ be the set of all (constrained) feasible payoffs that players can attain by employing arbitrary decision rules f. In passing, we note that $V(\delta, p^1) \subseteq \mathbb{R}^2$ is convex and compact.

It is important to observe that our definition of decision rules and set of feasible payoffs differ from those encountered in studies of stochastic games (Dutta 1995, Hörner, Sugaya, Takahashi, and Vieille 2010) and repeated games with incomplete information and communication (Escobar and Toikka 2013, Hörner, Takahashi, and Vieille 2015). Our definition of feasible payoffs already takes into account the fact that players cannot communicate and therefore player 2 can only decide based on public information.

Our focus will be on equilibrium strategies and payoffs. Since any strategy profile $s = (s_i)_{i=1}^2$ induces a decision rule, we deduce $\mathcal{E}(\delta, p^1) \subseteq V(\delta, p^1)$ for all $\delta < 1$.

4 Analysis

We will characterize equilibrium play in two steps. In the first step, we provide a dynamic programming formulation for efficient decision rules. This characterization will employ some tools from undiscounted optimization problems with partially observed states. In the second step we show how such efficient rule can be approximated by equilibrium play of the infinitely repeated game.

4.1 Efficient Payoffs

A decision rule f is *efficient* if for some $\alpha \in \mathbb{R}^2_{++}$, f is a solution to

$$q(\alpha) = \max\{\alpha \cdot v(f) \mid f \text{ is a decision rule}\}. \tag{4.1}$$

We can construct an efficient payoff vector $v = v^{\alpha,\delta} = v^{\delta}(f^{\alpha,\delta}) \in \mathbb{R}^2$, where $f^{\alpha,\delta}$ solves (4.1). Since any such $v^{\alpha,\delta}$ solves the problem $\max\{\alpha \cdot v \mid v \in V(\delta, p^1)\}$, the set of efficient payoff vectors v that maximize payoffs given a direction $\alpha \in \mathbb{R}^2_{++}$ is convex.

To characterize efficient decision rules, we introduce some notation. Let $\Sigma_1 = \{\sigma_1 \colon \Theta \to A_1\}$ be a set of *controls* for player 1 and let $\Sigma = \Sigma_1 \times A_2$. Let $p \in \Delta(\Theta)$ be a belief about player 1's type given public information, and let $p(\theta)$ denote the θ-element of p. For $\sigma \in \Sigma$ and $p \in \Delta(\Theta)$, we define the vector of expected period utility $U(\sigma, p) \in \mathbb{R}^2$ as

$$U_1(\sigma, p) = \sum_{\theta \in \Theta} u_1(\sigma_1(\theta), \sigma_2, \theta)\ p(\theta)$$

and $U_2(\sigma, p) = \sum_{\theta \in \Theta} u_2(\sigma_1(\theta), \sigma_2) p(\theta)$. For $\alpha \in \mathbb{R}^2_{++}$, we consider the ex-ante weighted sum of period payoffs

$$U^{\alpha}(\sigma, p) = \alpha \cdot U(\sigma, p) = \sum_{i=1}^{2} \alpha_i\ U_i(\sigma, p)$$

given a control profile $\sigma \in \Sigma$ and beliefs $p \in \Delta(\Theta)$. We also define the Bayes operator $B(\cdot \mid \sigma_1, p, a_1) \in \Delta(\Theta)$ as

$$B(\theta' \mid \sigma_1, p, a_1) = \sum_{\{\theta \mid \sigma_1(\theta \mid p) = a_1\}} P(\theta' \mid \theta)\ \frac{p(\theta)}{\sum_{\{\hat{\theta} \mid \sigma_1(\hat{\theta}) = a_1\}} p(\hat{\theta})} \tag{4.2}$$

whenever $\sigma_1(\hat{\theta}) = a_1$ for some $\hat{\theta}_1$ such that $p(\hat{\theta}) > 0$. We interpret $B(\theta' \mid \sigma_1, p, a_1)$ as the probability player 2 assigns to $\theta^{t+1} = \theta'$ given that at the beginning of round t his belief about θ^t was p, player 1 uses the control $\sigma_1 = \sigma_1(\theta^t)$, and player 2 observed player i's action $a_1^t = a_1$.

For $\alpha \in \mathbb{R}^2_{++}$ consider the only solution to the dynamic programming problem

$$w^{\alpha,\delta}(p) = \max_{\sigma\in\Sigma} \left\{(1-\delta)U^{\alpha}(\sigma,p) + \delta \sum_{a_1\in A_1} w^{\alpha,\delta}\Big(B(\cdot \mid \sigma_1, p, a_1)\Big) \sum_{\theta\in\Theta,\sigma_1(\theta)=a_1} p(\theta)\right\} \tag{4.3}$$

for all $p \in \Delta(\Theta_i)$.[15] Similar to the analysis in Section 2, this equation captures how, given beliefs p, a control determines current expected payoffs and continuation beliefs. Take $\sigma^{\alpha,\delta}(\cdot \mid p)$ as the control profile attaining the maximum in (4.3) as a function of beliefs p. Any σ such that $\sigma(\cdot \mid p) \to \Sigma$, for $p \in \Delta(\Theta)$, will be a (Markov) *control rule*. Using the control rule $\sigma^{\alpha,\delta}$, we can construct a (non-randomized) decision rule $f = f^{\alpha,\delta}$ from $\sigma^{\alpha,\delta}$ by setting

$$f_1^t(a^1,\ldots,a^{t-1},\theta^1,\ldots,\theta^t,\chi^1,\ldots,\chi^t) = \sigma_1^{\alpha,\delta}(\theta^t \mid p^t)$$

and

$$f_2^t(a^1,\ldots,a^{t-1},\chi^1,\ldots,\chi^t) = \sigma_2^{\alpha,\delta}(p^t)$$

where p^t is the belief that player 2 has about θ^t at the beginning of t. Observe that the sequence $(p^t)_{t\geq 1}$ can be recursively computed as

$$p^{t+1}(\theta) = B(\theta \mid \sigma_1^{\alpha,\delta}(\cdot \mid p^t), p^t, a_1^t) \text{ for } t \geq 1,$$

given the initial belief p^1.

Lemma 4. *Let $\alpha \in \mathbb{R}^2_{++}$. The following hold:*

a. *The value of the maximization problem (4.1) is $q(\alpha) = w^{\alpha,\delta}(\lambda)$.*

b. *The decision rule $f = f^{\alpha,\delta}$ constructed from $\sigma^{\alpha,\delta}$ above is a solution to (4.1).*

Like most of the literature in repeated games (Fudenberg and Maskin 1986, Athey and Bagwell 2008, Hörner, Sugaya, Takahashi, and Vieille 2011), we will characterize equilibrium behavior when players are patient. It will be useful to consider efficient decision rules and payoffs as $\delta \to 1$. We define the *differential discounted value* function as

$$h^{\alpha,\delta}(p) = \frac{w^{\alpha,\delta}(p)}{1-\delta} - \frac{w^{\alpha,\delta}(p^1)}{1-\delta} \tag{4.4}$$

for any $p \in \Delta(\Theta)$. Using this definition we can rewrite (4.3) as

$$h^{\alpha,\delta}(p) + w^{\alpha,\delta}(p^1) = \max_{\sigma\in\Sigma} \Big\{U^{\alpha}(\sigma,p) + \delta \sum_{a_1\in A_1} h^{\alpha,\delta}\big(B(\cdot \mid \sigma_1, p, a_1)\big)\big(\sum_{\theta\in\Theta,\sigma_1(\theta)=a} p(\theta)\big)\Big\} \tag{4.5}$$

[15]The existence and uniqueness of the solution $w^{\alpha,\delta}$ follows from standard arguments, see Stokey and Lucas (1989).

Just to set ideas, assume that there exist subsequences $(h^{\alpha,\delta^\nu})_{\nu\geq 0}$ and $(w^{\alpha,\delta^\nu})_{\nu\geq 0}$ pointwise converging to functions $h^\alpha\colon \Delta(\Theta)\to\mathbb{R}$ and $w^\alpha\colon \Delta(\Theta)\to\mathbb{R}$. That is, $h^\alpha(p)=\lim_{\nu\to\infty} h^{\alpha,\delta^\nu}(p)$ and $w^\alpha(p)=\lim_{\nu\to\infty} w^{\alpha,\delta^\nu}(p)$ for all p. Therefore $\rho^\alpha=\lim_{\nu\to\infty} w^{\alpha,\delta^\nu}(p^1)$ does not depend on p^1. Taking the limit in equation (4.5), we deduce that the pair $(h,\rho)=(h^\alpha,\rho^\alpha)$ solves the *average reward optimality equation* (AROE)

$$h(p)+\rho=\max_{\sigma\in\Sigma}\Big\{u^\alpha(\sigma,p)+\sum_{a_1\in A_1} h\big(B(\cdot\mid\sigma_1,p,a_1)\big)\big(\sum_{\theta\in\Theta,\sigma_1(\theta)=a_1} p(\theta)\big)\Big\} \tag{4.6}$$

for all $p\in\Delta(\Theta)$. Let $\sigma^\alpha(\cdot\mid p)\in\Sigma$ be the control profile attaining the maximum in the dynamic programming problem (4.6) given $p\in\Delta(\Theta)$.

The following result establishes the key properties connecting the discounted and undiscounted dynamic programing problems.

Theorem 1 (Optimality Theorem). *Fix $\alpha\in\mathbb{R}^2_{++}$. The following hold:*

a. *The AROE (4.6) has a solution (h^α,ρ^α) and a control rule σ^α that attains the optimum in (4.6).*

b. *For any converging subsequence $h^{\alpha,\delta^\nu}\to\bar{h}$ as $\nu\to\infty$, we can take $\rho=\lim_{\nu\to\infty} w^{\alpha,\delta^\nu}(p^1)$ that does not depend on p^1, and obtain a pair $(\bar{h},\rho)$ that solves the AROE (4.6). The function $\bar{h}\colon\Delta(\Theta)\to\mathbb{R}$ is convex.*

c. *For any decision rule f, $\limsup_{\delta\to 1}\sum_{i=1}^2\alpha_i v_i^\delta(f)\leq\rho=\lim_{\nu\to\infty} w^{\alpha,\delta^\nu}(p^1)$.*

This result shows that studying (4.6) is useful to determine optimal payoffs and behavior when players are patient. The first part ensures existence of solution. This is not obvious since (4.6) does not define a contraction map. The second part shows that such solution can be obtained by solving problems with discount factores that go to 1. The third part formally establishes that the solution $\rho\in\mathbb{R}$ to (4.6) provides a tight upper bound for the value of the discounted problem, as the discount factor goes to 1.

The AROE (4.6) is central to our analysis. The right-hand side of (4.6) captures the trade-off that an optimal control σ solves as a function of current beliefs $p\in\Delta(\Theta)$. An optimal rule takes into account current period payoffs and the distribution over beliefs in the subsequent round. Since the differential value h is convex in $p\in\Delta(\Theta)$, more precise beliefs always improve continuation payoffs. In particular, when rules that separate types maximize current weighted payoffs, they also maximize total undiscounted weighted payoffs.

Proposition 1. *Consider a belief $p\in\Delta(\Theta)$ and a rule $\bar{\sigma}=(\bar{\sigma}_1,\bar{\sigma}_2)$ with $\bar{\sigma}_1\colon\Theta\to A_1$ and $\bar{\sigma}_2\in A_2$ such that for all $\theta\neq\theta'$, $\bar{\sigma}_1(\theta)\neq\bar{\sigma}_1(\theta')$ and*

$$\bar{\sigma}\in\arg\max_{\sigma\in\Sigma} U^\alpha(\sigma,p).$$

Then,

$$\bar{\sigma} \in \arg\max_{\sigma \in \Sigma} \Big\{ U^{\alpha}(\sigma, p) + \sum_{a_1 \in A_1} h\big(B(\cdot \mid \sigma_1, p, a_1)\big)\big(\sum_{\theta \in \Theta, \sigma_1(\theta) = a_1} p(\theta)\big)\Big\}. \tag{4.7}$$

Observe that the fully separating rule $\bar{\sigma}$ in the Proposition need not be optimal for (4.7). The benefit of a separating rule like $\bar{\sigma}$ in (4.7) is that player 1 can use his private information to maximize his own ex-post total payoffs, while the cost is that player 2's payoff is maximized when player 1's behavior can be perfectly predicted.[16] More generally, solutions to (4.6) will involve a complex mix of trade-offs, and explicit formulas are in general unfeasible.[17]

4.2 Equilibrium Behavior

In this section, we investigate the conditions under which the undiscounted optimal dynamics can be approximated by an equilibrium of our repeated game.

A control rule σ together with the initial beliefs p^1 recursively determine a belief process $(p^t)_{t \geq 1}$ by

$$p^{t+1} = B(\cdot \mid \sigma, p^t, a_1^t) \quad \forall t \geq 1.$$

Given any control rule σ, the joint process $(\theta^t, p^t)_{t \geq 1}$ is Markovian, with p^1 and θ^1 given.

Definition 1. *A control rule σ determines* a unique recurrence class *if the process $(\theta^t, p^t)_{t \geq 1}$ is a finite Markov chain having a unique recurrence class.*[18]

This is arguably an important restriction over the Markov process $(\theta^t, p^t)_{t \geq 1}$. On the one hand, the path of the Markov chain $(\theta^t, p^t)_{t \geq 1}$ could be countable. This case arises when the rule pools all types along the path and the initial belief does not coincide with the stationary distribution of P. Exploring the ergodicity properties of $(\theta^t, p^t)_{t \geq 1}$ in hidden Markov models is a question dating back to Blackwell (1951). Interesting recent developments exist, but they do not apply to a model like ours in which the observation variable is endogenous.[19]

More generally, several results in the literature restrict attention to models in which ergodicity restrictions on state variables are (directly or indirectly) imposed (Dutta 1995, Hörner, Sugaya, Takahashi, and Vieille 2011, Renault, Solan, and Vieille 2013). While checking

[16] The solution to the maximization problem $\max_{\sigma \in \Sigma} U_2(\sigma, p)$ will typically be a pooling rule, in which player 2 can perfectly predict the action player 1 will employ.

[17] Problem (4.6) is similar to a bandit problem with Markovian hidden state (Keller and Rady 1999). Separating rules maximize *exploration*. Propositions 1 and 3 show conditions under which the standard exploration vs exploitation dilemma (Bergemann and Valimaki 2006) does not arise.

[18] In other words, a control rule determines a unique recurrence class it if there exists a finite set $\mathcal{P} \subseteq \Delta(\Theta)$ such that $(\theta^t, p^t)_{t \geq 1} \subseteq \Theta \times \mathcal{P}$ and a unique subset $\mathcal{P}' \subseteq \mathcal{P}$ such that for all $(\theta, p) \in \Theta \times \mathcal{P}'$, if the Markov chain visits (θ, p), then in the next period it will stay in $\mathcal{P}'$ with probability 1, and no proper subset of $\mathcal{P}'$ has this property. See Stokey and Lucas (1989) for additional discussion.

[19] Recent results by Van Handel (2009) and Tong and Van Handel (2012) apply to models in which the observation variable (the action in our case) is assumed to have full support. To adapt those results to our setup one would need to restrict attention to perfectly mixed controls, which are suboptimal in our model. While it is true that an optimal control rule can be approximated by perfectly mixed controls, the process of beliefs would have an infinite support. Our proof for Theorem 2 cannot be extended to accommodate infinite beliefs.

whether an optimal rule determines a unique recurrence class Markov requires a characterization of solutions to (4.6), such exercise is easy to execute in many applications (see Section 5). In particular, the following result is useful in applications.

Proposition 2. *Assume that the control rule σ is such that for any belief $p \in \Delta(\Theta)$ having positive probability in the path $(\theta^t, p^t)_{t\geq 1}$, types are separated: $\sigma(\theta \mid p) \neq \sigma(\theta' \mid p)$ for all $\theta \neq \theta'$. Then, σ determines a unique recurrence class.*

This result follows since when types are separated, continuation beliefs come from the set $\{P(\cdot \mid \theta) \mid \theta \in \Theta\}$. In this case, the support of the process $(\theta^t, p^t)_{t\geq 1}$ is $\Theta \times (\{p^1\} \cup \{P(\cdot \mid \theta) \mid \theta \in \Theta\})$ and its only recurrence class is $\Theta \times \{P(\cdot \mid \theta) \mid \theta \in \Theta\}$.

We will also consider rules where all types pool.

Definition 2. *A control rule σ* determines a pooling path of actions *if there exists an action $a \in A$ such that*

$$\mathbb{P}_\sigma[\sigma(\theta^t \mid p^t) = a, \forall t \geq 1] = 1$$

Observe that for any control rule σ determining a unique recurrence class or a pooling path of actions, the limit-average payoffs

$$v_1^\infty(\sigma) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\sum_{t=1}^{T} u_1(\sigma(\theta^t \mid p^t), \theta^t)], \quad v_2^\infty(\sigma) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\sum_{t=1}^{T} u_2(\sigma(\theta^t \mid p^t))]$$

are well defined. This is obvious if the rule is pooling. For a control rule σ determining a unique recurrence path, this follows from Proposition 8.1.1 in Puterman (2005) after noticing that the limits are average rewards from a stationary Markov decision rule over a finite state Markov process. Letting $\bar{\pi} = \bar{\pi}^\sigma \in \Delta(\Theta \times \mathcal{P})$ be the stationary distribution for the Markov chain $(\theta^t, p^t)_{t\geq 1}$, given the control rule σ, with $\Theta \times \mathcal{P}$ the recurrence class of the chain, it follows that

$$v_1^\infty(\sigma) = \sum_{(\theta,p)\in\Theta\times\mathcal{P}} u_1(\sigma(\theta \mid p), \theta)\bar{\pi}(\theta, p) \quad \text{and} \quad v_2^\infty(\sigma) = \sum_{(\theta,p)\in\Theta\times\mathcal{P}} u_2(\sigma(\theta \mid p))\bar{\pi}(\theta, p)$$

We define $v^\infty(\sigma) = (v_i^\infty(\sigma))_{i=1,2}$.

The main purpose of this subsection is to find conditions under which payoffs generated using different control rules can be attained. This is relatively simple when the control rule determines a pooling path as in this case deviations are immediately observed. But the problem is much more complicated for control rules in which player 1 is expected to use his private information.

Fix a control rule σ determining a unique recurrence class $\Theta\times\mathcal{P}$. Define $m_1^\sigma(\cdot \mid p) \in \Delta(A_1)$ as the distribution over actions given a belief $p \in \mathcal{P}$ by

$$m_1^\sigma(a_1 \mid p) = \sum_{\{\theta\in\Theta \mid a_1=\sigma_1(\theta\mid p)\}} p(\theta)$$

For $a \in A$ and $p \in \mathcal{P}$, we define $m^\sigma(a \mid p)$ analogously.

Given any sequence of actions $a_1^1, \ldots, a_1^t$ and a fixed control rule σ determining an irreducible Markov chain, we can mechanically calculate probabilities $\bar{p}^{t+1} = B(\cdot \mid \sigma_1, \bar{p}^t, a_1^t)$ (if this is not well defined, we set $\bar{p}^{t+1}$ to be an arbitrary element of the support of the process of beliefs $(p^t)_{t\geq 1}$). These *simulated probabilities* need not coincide with the beliefs a Bayesian agent would have about current types as player 1's actions in the game could be derived from an arbitrary strategy s_1. We will sometimes emphasize the dependence of $(\bar{p}^t)_{t\geq 1}$ on the control rule σ by writing $(\bar{p}^t(\sigma))_{t\geq 1}$. For a control rule σ determining a unique recurrence class with support $\Theta \times \mathcal{P}$ and given any history $(a^t, \theta^t, \bar{p}^t(\sigma))_{t\geq 1}$, we can compute the occupancy rate of actions conditional on simulated probabilities as

$$\bar{m}^\delta(a \mid p) = \frac{\sum_{t=1}^\infty \delta^{t-1} \mathbb{1}_{\{a^t = a, \bar{p}^t = p\}}}{\sum_{t=1}^\infty \delta^{t-1} \mathbb{1}_{\{\bar{p}^t = p\}}}$$

for each $a \in A$ and $p \in \mathcal{P}$.

We define the stationary minmax value as the smallest payoff a player can attain when his rival chooses a fixed action and he chooses actions optimally. More formally,

$$\underline{v}_1 = \min_{a_2 \in A_2} \mathbb{E}_\pi[\max_{a_1 \in A_1} u_1(a, \theta)], \quad \underline{v}_2 = \min_{a_1 \in A_1} \max_{a_2 \in A_2} u_2(a).$$

Our minmax definition does not yield the lowest payoff one could consider against a player (Escobar and Toikka 2013, Hörner, Takahashi, and Vieille 2015), but it is simple to work with and fully satisfactory in many applications. A vector $v \in \mathbb{R}^2$ is *strictly individually rational* if $v_i > \underline{v}_i$ for $i = 1, 2$.

The following theorem shows that the optimality analysis performed in Section 4.1 is useful to understand optimal equilibrium behavior.

Theorem 2 (Equilibrium Theorem)**.** *Let σ^α be a control rule solving (4.6) given weights $\alpha \in \mathbb{R}_{++}$, determining a unique recurrence class $\Theta \times \mathcal{P}$, and resulting in a payoff vector $v = v^\infty(\sigma^\alpha) \in \mathbb{R}^2$. Assume*

i. There exists control rules σ^1, σ^2 such that for each i either (a) σ^i determines a pooling path of actions or (b) σ^i solves (4.6) given some weights $\alpha^i \in \mathbb{R}_{++}$ and determines a unique recurrence class;

ii. All payoff vectors $v, v^1 \equiv v^\infty(\sigma^1), v^2 \equiv v^\infty(\sigma^2)$ are strictly individually rational;

iii. $v_i^i < v_i < v_i^{-i}$, for $i = 1, 2$.

Then, for all $\epsilon > 0$ there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, the infinitely repeated game with discount factor δ has a perfect Bayesian equilibrium $s^* = (s_1^*, s_2^*) \in \mathcal{E}(\delta, p^1)$ such that

a. After all on-path histories, expected continuation values are within ϵ of v

b. $\mathbb{P}_{s^*}\Big[\max_{a\in A, p\in\mathcal{P}}|\bar{m}^{\delta}(a \mid p) - m^{\sigma^{\alpha}}(a \mid p)| < \epsilon\Big] \geq 1-\epsilon.$

This result characterizes approximately optimal equilibrium behavior. It shows that provided players are patient enough, players' incentives can be aligned to attain payoffs close to $v \in \mathbb{R}^2$. Moreover, with sufficiently high probability, conditional on simulated beliefs, players equilibrium actions will approximate the frequencies induced by the optimal rule σ^{α}. In other words, given observed actions, equilibrium behavior cannot be distinguished from optimal dynamics.[20]

The proof of Theorem 2 proceeds by constructing strategies in which player 2 forgives but does not forget. To do that, we revisit the review strategy idea by Radner (1981) and Townsend (1982) and build strategies in which player 2 keeps checking whether player 1's actions can be distinguished from the efficient rule σ_1^{α}. The details of our formulation are closely related to the quota mechanisms in Jackson and Sonnenschein (2007), Renault, Solan, and Vieille (2013), and particularly Escobar and Toikka (2013). One conceptual difference is that in our model players cannot explicitly communicate and therefore we cannot formulate the problem as a mechanism design one.

For a given sequence of actions $(a^1, \ldots, a^t) \in A^t$ and $(p, a_1) \in \mathcal{P} \times A_1$, define

$$N^t(p) = \sum_{t'=1}^{t} \mathbb{1}_{\{\bar{p}^{t'}=p\}}, \quad N^t(p, a_1) = \sum_{t'=1}^{t} \mathbb{1}_{\{(\bar{p}^t, a_1^t)=(p,a_1)\}}$$

and let

$$\bar{m}^t(a_1 \mid p) = \frac{N^t(p, a_1)}{N^t(p)}.$$

be the empirical frequency of player 1's actions conditional on the simulated probability being $\bar{p}^t = p$.

The first step is to introduce an artificial player, player 0, who decides player 2's actions (as a function of the history) and can also decide whether he lets player 1 to choose his action or whether player 0 itself chooses player 1's action at any given round. We will fix the behavior of player 0 and analyze the incentives player 1 has when deciding actions. Player 0 can always "interpret" an action by player 1 through the control $\sigma_1^{\alpha}(\cdot \mid \bar{p}^t)$ given the *simulated* probability $\bar{p}^t$. Indeed, given a history of actions $(a^1, \ldots, a^t)$, player 0 computes the simulated probabilities $\bar{p}^1 = p^1$ and recursively define $\bar{p}^{t+1}(\cdot) = B(\cdot \mid \sigma_1^{\alpha}, p^t, a_1^t)$. If $a_1^t \notin A_1(\bar{p}^t)$, we assign $\bar{p}^{t+1} = p_0$ where $p_0 \in P$ is arbitrary.

For any decreasing sequence (b_k) converging to 0, we say that player 1 *passes the test* (b_k) given a history $(a^1, \ldots, a^t) \in A^t$ if

$$\max_{a_1 \in A_1} |m_1(a_1 \mid p) - m_1^t(a_1 \mid p)| \leq b_t$$

[20]In contrast to two-player repeated games with complete information, our result requires the existence of player-specific punishments (Fudenberg and Maskin 1986). In our problem, types are hidden and for some types the minmaxing action could actually yield high payoffs to the minmaxed player.

for all $p \in \mathcal{P}$. Given $T \geq 1$, a rule σ^α and sequence (b_k), the game of credible play $(\sigma^\alpha, (b_k), T)$ is constructed as follows. For $t \leq T$, if player 1 has passed the test (b_k) in all previous rounds $t' = 1, \ldots, t-1$, then he can freely select his action a_1^t; otherwise, player 0 chooses a_1^t by randomly drawing an action according to the distribution $m(\cdot \mid \bar{p}^t)$. We define the *obedient strategy* for player 1 as $\hat{s}_1^t(\theta^1, \ldots, \theta^t, a^1, \ldots, a^{t-1}) = \sigma_1^\alpha(\theta^t \mid \bar{p}^t)$ whenever he is allowed to choose actions. We will also define the block-game of credible player $(\sigma^\alpha, (b_k), T)^\infty$ as the infinite horizon problem in which a game of credible play restarts after T rounds of play (with discount factor δ).

Lemma 5. *Let* $\epsilon > 0$.

a. There exists a test (b_k) *such that, for any initial belief* $p^1 \in \Delta(\Theta)$

$$\mathrm{P}_{\hat{s}_1}[\textit{Player 1 passes the test } (b_k) \textit{ at } (a^1, \ldots, a^t) \textit{ for all } t] \geq 1 - \epsilon.$$

when player 1 uses the obedient strategy $\hat{s}_1$.

b. There exists a test (b_k) *and* $\bar{\delta} < 1$ *such that for all* $\delta > \bar{\delta}$ *there exists* $\bar{T}$ *such that for all* $T \geq \bar{T}$, *for any sequential best response* s_1 *of Player 1 in the block-game of credible play* $(\sigma^\alpha, (b_k), T)^\infty$ *given discount* δ, *the vector of discounted period payoffs* $v^\delta(s_1) \in \mathbb{R}^2$ *is within distance* ϵ *of* v. *Moreover,*

$$\mathbb{P}_{s_1}\Big[\max_{a_1 \in A_1, p \in \mathcal{P}} |\bar{m}^\delta(a_1 \mid p) - m^{\sigma^\alpha}(a_1 \mid p)| < \epsilon\Big] \geq 1 - \epsilon$$

To establish Theorem 2, we use this lemma to construct strategies delivering the desired equilibrium payoffs v^α. Strategies are of the stick-and-carrot type (Fudenberg and Maskin 1986). On the path of play, players choose actions mimicking the path of play in the equilibrium of the block-game of credible play from Lemma 5. Any observable deviation by i triggers a punishment phase, in which player i is minmaxed by a number of rounds, and then play proceed to a carrot phase in which players mimic the play of the game of credible play yielding payoffs v^i.

5 Games with Separating and Monotonic Dynamics

We now provide a characterization of solutions to (4.6). We assume that A_1 and Θ are contained in $\mathbb{R}$ and write $A_1 = \{a^n \mid n = 1, \ldots, |A_1|\}$ and $\Theta = \{\theta^m \mid m = 1, \ldots, |\Theta|\}$ with $a^n < a^{n+1}$ and $\theta^m < \theta^{m+1}$. We extend the payoff function for player 1, u_1, to actions $a_1 \in \mathbb{R}$ and states $\theta \in \Theta$ so that $u_1(a_1, a_2, \theta)$ is twice continuously differentiable in $(a_1, \theta) \in \mathbb{R} \times \mathbb{R}$.

Definition 3. *We will say that* u_1 *has* strongly increasing differences *in* (a_1, θ) *if*

$$\min\left\{\frac{\partial^2 u_1(a_1, a_2, \theta)}{\partial a_1 \partial \theta} \mid a_1 \in \mathbb{R}, a_2 \in A_2, \theta \in \mathbb{R}\right\} > 0.$$

The following result shows conditions under which the optimal control rule is strictly increasing. Since actions are discrete, this property cannot be inferred by simply appealing to strong increasing differences. There are two forces behind this result. Separating rules (in particular, strictly increasing rules) make continuation beliefs more precise and therefore maximize continuation payoffs (Proposition 1). This effect is reinforced when the action set is rich because in this case the maximization of total period payoffs yield strictly increasing rules.

Proposition 3. *Assume that u_1 has strongly increasing differences in (a_1, θ). Let $\alpha \in \mathbb{R}^2_{++}$ be such that*

$$\alpha_1 u_1(a^{|A_1|-1}, a_2, \theta) + \alpha_2 u_2(a^{|A_1|-1}, a_2) > \alpha_1 u_1(a^{|A_1|}, a_2, \theta) + \alpha_2 u_2(a^{|A_1|}, a_2) \tag{5.1}$$

and

$$\alpha_1 u_1(a^1, a_2, \theta) + \alpha_2 u_2(a^1, a_2) < \alpha_1 u_1(a^2, a_2, \theta) + \alpha_2 u_2(a^2, a_2) \tag{5.2}$$

for all $a_2 \in A_2$ and all $\theta \in \Theta$ and $\alpha_1 u_1(a_1, a_2, \theta) + \alpha_2 u_2(a_1, a_2)$ is concave in $a_1 \in \mathbb{R}$. Define

$$c_1 = \max_{a_1 \in \mathbb{R}, a_2 \in A_2, \theta \in \mathbb{R}} \Big(-\alpha_1 \frac{\partial^2 u_1(a_1, a_2, \theta)}{\partial a_1^2} - \alpha_2 \frac{\partial^2 u_2(a_1, a_2)}{\partial a_1^2} \Big) \geq 0$$

and

$$c_2 = \min_{a_1 \in \mathbb{R}, a_2 \in A_2, \theta \in \mathbb{R}} \frac{\partial^2 u_1(a_1, a_2, \theta)}{\partial a_1 \partial \theta} > 0.$$

Assume that

$$\frac{2c_1}{\alpha_1 c_2} \max_{n=1,\dots,|A_1|-1} \{a^{n+1} - a^n\} < \min_{m=1,\dots,|\Theta|-1} \{\theta^{m+1} - \theta^m\}. \tag{5.3}$$

Then, any rule σ^α attaining the maximum in (4.6) is such that $\sigma_1^\alpha(\theta \mid p)$ is strictly increasing as a function of θ for all $p \in \Delta(\Theta)$ with $p(\theta) > 0$ for all $\theta \in \Theta$. Moreover, when the transition matrix P is such that $P(\cdot \mid \theta')$ first order stochastically dominates $P(\cdot \mid \theta)$ for all $\theta' \geq \theta$, and $u_1(a, \theta)$ and $u_2(a)$ are supermodular (in (a, θ) and a respectively), then $\sigma^\alpha(\theta \mid p)$ is nondecreasing in (θ, p), where P is endowed with the (partial) order given by first order stochastic dominance.

Under the conditions above, the separating rule σ^α determines a unique ergodic class therefore Theorem 2 can readily be applied. Some applications follow.

5.1 Collusion with Bertrand Competition

Tacit collusion is a prominent feature of many industries, as documented, for example, by Bresnahan (1987) for the American automobile market, and by Blume, Strand, and Färnstrand (2002) for the European industrial sugar market. In this section, we study a model of tacit collusion with Bertrand competition. Two firms set prices $a_i \in A_i$ at each $t = 1, 2 \dots$. Firms sell heterogeneous goods, and each firm i faces a demand $Q_i(a_i, a_j)$ which is decreasing in a_i

and increasing in a_j. Firms have constant returns to scale. While firm 2's costs are known to equal $c > 0$, firm 1's costs are private information $\theta \in \{\underline{\theta}, \bar{\theta}\}$, with $\underline{\theta} < \bar{\theta}$. Players' utility functions take the form

$$u_1(a_1, a_2, \theta) = Q_1(a_1, a_2)(a_1 - \theta)$$

and

$$u_2(a_1, a_2) = Q_2(a_1, a_2)(a_2 - c).$$

where $Q_i(a_i, a_{-i}) = (1 - a_i + za_{-i})$ with $z \in]0, 1[$. We assume that types follow a Markov chain P with $P(\theta' \mid \theta) > 0$ for all $\theta', \theta \in \{\underline{\theta}, \bar{\theta}\}$, with $P(\bar{\theta} \mid \bar{\theta}) > P(\bar{\theta} \mid \underline{\theta})$.

We can apply Proposition 3 to characterize the welfare maximizing control rule σ^α, for $\alpha = (1, 1)$. Under the interiority restrictions (5.1)- (5.2), the welfare maximizing rule σ^α will be separating provided $\max\{a^{n+1} - a^n\} < \frac{1}{4}(\bar{\theta} - \underline{\theta})$. Up to integer restrictions,

$$\sigma_2^\alpha(p) = \frac{1}{2}\Big\{\frac{1}{1+z}(c - E_p[\theta]) + \frac{1}{1-z}\Big\}$$

and

$$\sigma_1^\alpha(\theta \mid p) = \frac{1}{2}\Big\{\theta + 1 + 2z\sigma_2^\alpha(p) + c\Big\}$$

Under the optimal control rule σ^α, firm 1 signals his type by choosing a higher price when his cost is high. When firm 1 chooses a high price in period t, then his cost is more likely to be high in period $t + 1$ and player 2's price is also higher. In this sense, a low price by firm 1 in t triggers a price war in $t + 1$, in which firm 2's price is low and firm 1's prices are also low. The severity of the price war (i.e. how low prices will be) will depend on the gap $\bar{\theta} - \underline{\theta}$ and on how persistent the low cost state $\bar{\theta}$ is. The price war is over only once firm 1's price raises. Observe that σ^α is a rule determining a unique recurrence class and therefore Theorem 2 applies.

As Marshall and Marx (2013) explain, during the period 2000-2005, the European Commission classified 9 out of the 22 major industrial cartels as showing evidence of "of frequent bargaining problems and deviations by cartel members, occurring throughout the cartel period."[21] These "deviations" are also highlighted by Genesove and Mullin (2001) in the study of the sugar cartel. In contrast to other theoretical papers, in our setup equilibrium price cuts actually occur and apparent deviations can be seen as the result of firms using their private information to maximize total profits and signaling their continuation play.[22]

Our model also explains collusive price leadership: the informed firm becomes a price leader as whenever it raises its price in t, firm 2's price will be higher in $t + 1$. Thus, our model gives theoretical support to Stigler's (1947) observation that price leadership may be an efficient mechanism to transmit information, and to Markham's (1951) view that firms

[21] See also Bernheim and Madsen (2014).

[22] Models of price dispersion could also be interpreted as generating equilibrium price cuts. See Bernheim and Madsen (2014) for an application of this idea in a collusion context. Price cuts could also improve monitoring in collusion models with imperfect public monitoring (Rahman 2014).

may use “price leadership in lieu of an overt agreement.” Several papers document instances of collusive price leadership. Allen (1976) shows evidence of collusive price leadership in the market of steam turbine generators, and Marshall, Marx, and Raiff (2008) discuss evidence of collusive price leadership in vitamins, rubber chemicals, sorbates, monochloroacetic acid and organic peroxides, polyester staple, high-pressure laminates, amino acids, carbonless paper, cartonboard, and graphite electrodes. Mouraviev and Rey (2011) show that price leadership features in 16 out of 49 European Commission’s cartel decisions as of July 2010. Rotemberg and Saloner (1990) also study collusion and price leadership in a Bertrand model with incomplete information. Their model exhibits iid private information and for price leadership to emerge, within each round the informed firm must set its price before the uninformed one. Such sequentiality is not needed in our model.

Our collusion model differs from the more standard analysis of Bertrand games with inelastic demand and incomplete information about costs. In Athey and Bagwell (2001) firms have iid private costs and, before choosing actions, can freely exchange messages. Athey and Bagwell (2008) and Escobar and Toikka (2013) extend the model to allow for serially dependent private costs. In all these works, firms can be arbitrarily close to the first best collusive outcome, in which only the lowest cost firm produces and fixed the consumers’ reservation value. As Athey and Bagwell (2001) observe, communication can be dispensed with as prices can be used to signal costs (at an arbitrarily low cost). But this observation crucially depends on the assumption of *inelastic demand*. Our analysis shows that in more general Bertrand games, firms are bounded away from a perfectly collusive outcome when the exchange of messages is costly.[23] More importantly, in the standard Bertrand models of Athey and Bagwell (2001) and Escobar and Toikka (2013), the path of collusive prices cannot be distinguished from the prices one would observe when firms’ information is symmetric (as in Rotemberg and Saloner 1986). In contrast, our analysis not only shows that the costs of incomplete information can be substantive for a cartel, but also that asymmetric information has nontrivial implications for the dynamics of prices.[24]

5.2 Graduated Sanctions in Collective Action Games

Case studies show that punishments are not drastic but rather gradual. In many groups, players “who violate operational rules are likely to be assessed graduated sanctions” (Ostrom 1990, p. 94) and are even given opportunities to make restitutions. As Dixit (2009) and Abreu, Bernheim, and Dixit (2005) argue, this evidence contrasts with the more standard theories of repeated games with perfect and imperfect monitoring (Abreu 1988, Green and Porter 1984, Abreu, Pearce, and Stacchetti 1986, Abreu, Milgrom, and Pearce 1991). Our set-up provides a rational for graduated sanctions in repeated games based on incomplete

[23] The results in Escobar and Toikka (2013) show that firms can attain perfectly collusive outcomes if they communicate.

[24] Athey, Bagwell, and Sanchirico (2004) study a repeated Bertrand game with iid cost and show that optimal equilibrium is in (on-path) pooling strategies when firms are restricted to use strongly symmetric strategies.

information.[25]

We specialize our model to a repeated collective action game. At each t, players simultaneously choose actions $a_i \in A_i \subseteq \mathbb{R}_{++}$. Player 1's type θ belongs to a set $\Theta \subset \mathbb{R}_{++}$. An action is interpreted as a contribution to the team (or group), whereas player 1's private type determines how much player 1 benefits from the contributions. More concretely, we assume that

$$u_1(a, \theta) = \theta a_1 a_2 - a_1^2 \quad \text{and} \quad u_2(a, \theta) = a_1 a_2 - a_2^2.$$

The terms $-a_i^2$ in the utility functions capture the costs of contributing, while $\theta a_1 a_2$ and $a_1 a_2$ are the benefits obtained by each player from the complementary contributions. The benefit that player 1 obtains from the project is privately known. We assume that the transition $P(\cdot \mid \theta)$ first-order stochastically increases in θ.

Proposition 3 implies that, under the boundary restrictions and $\max\{a^{n+1} - a^n\} \leq 4 \min\{a_2 \in A_2\} \min\{\theta^{m+1} - \theta^m\}$, the rule maximizing the sum of utilities $\sigma^* = (\sigma_1(\theta \mid p), \sigma_2(p))$ is separating and increases in (θ, p). Given a belief $p \in \Delta(\Theta)$, consider two actions for player 1 $\bar{a}_1 > \hat{a}_1$. the corresponding continuation beliefs are $\bar{p} > \hat{p}$. This means that the actions that player 2 will take are, respectively, $\bar{a}_2 > \hat{a}_2$. More generally, conditional on p^t, the action chosen by player 2 in $t+1$ is strictly increasing as a function of the action chosen by player 1 in t, a_1^t. This means that the dynamics generated in our model of incomplete information exhibit graduated sanctions (players always contribute positive amounts) that fit the size of previous contributions. Player 1 also "makes restitutions" by taking higher actions that positively affect player 2's continuation beliefs and actions.

6 Equilibrium as Interactions Become Frequent

Our limit results, Theorems 1 and 2, apply when $\delta \to 1$. As Abreu, Milgrom, and Pearce (1991) point out, the limit $\delta \to 1$ can be interpreted saying that either interest (discount) rates are low or that players move frequently. In games with imperfect monitoring, Abreu, Milgrom, and Pearce (1991) show that the two interpretations can lead to radically different results as when moves become more frequent not only the interest rates change but also the quality of the monitoring technology. In our perfect monitoring game of incomplete information, the impact of more frequent moves is also subtle as types are more likely to remain unchanged between two consecutive rounds. In this subsection, we explore these issues in a simple prisoners' dilemma.

Two players choose actions at each $t = D, 2D, \ldots$, where $D > 0$ is the period length. At each t, players play a game as in Section 2, with the payoffs given in Table 1. Monitoring is perfect, but only player 1 can observe $\theta^t \in \{l, h\}$ at the beginning of round t, with $l < h$. We parameterize both the discount factor and the transitions by D. The discount factor equals

[25] Other theories that can explain graduated sanctions come from repeated extensive-form games (Mailath, Nocke, and White 2004).

$\delta = \exp(-rD)$, where $r > 0$ is the discount rate per time unit. Transitions are given by

$$\mathbb{P}[\theta^t = l \mid \theta^{t-1} = l] = 1 - \phi D$$

whereas

$$\mathbb{P}[\theta^t = h \mid \theta^{t-1} = h] = 1 - \chi D$$

with $\phi, \chi > 0$. The initial type is drawn from a probability distribution such that $\mathbb{P}(\theta^1 = l) = 1 - \phi D$.

We explicit the dependence of the transition matrix and the Bayes operator on D by writing $P = P^D$ and $B = B^D$. Under this parametrization we can interpret our findings in Section 2 as taking the interest rate to 0 ($r \to 0$). Our interest now is in the limit $D \to 0$.

The formulation of the dynamic programming problem characterizing decision rules that maximize the sum of payoffs for $D > 0$ can be imported from Sections 2 and 4. More explicitly, given a belief $p = \mathbb{P}[\theta^t = l]$, the value function for the problem of maximizing the sum of payoffs is

$$\begin{aligned} w^D(p) \quad = \quad & \max_{\sigma \in \Sigma} \Bigg\{ (1 - \exp(-rD))\, U^{(1,1)}(\sigma, p) \\ & + \exp(-rD) \sum_{a_1 \in \{I,N\}} w^D\big(B^D(\cdot \mid \sigma_1, p, a_1)\big) \sum_{\theta, \sigma_1(\theta) = a_1} p(\theta) \Bigg\}. \end{aligned} \tag{6.1}$$

The following result characterizes the solution to this problem when D is small.

Proposition 4. *The following hold:*

a. *There exists $\bar{D} > 0$ such that for all $D < \bar{D}$ and all $p \in [\chi D, 1 - \phi D]$, the right-hand side of (6.1) has a unique solution $\bar{\sigma}$. Such solution is such that $\sigma_1(l \mid p) = I$ and $\sigma_1(h \mid p) = NI$. Moreover, $w^D(p) \to 2(a - l)\frac{\chi}{\phi + \chi}$ as $D \to 0$.*

b. *For all $\epsilon > 0$, there exists $\hat{D} \in]0, \bar{D}[$ such that for $D < \hat{D}$ we can find $\bar{r}(= \bar{r}(D)$ such that the game played every D units of time with discount rate $r < \bar{r}(D)$ has an equilibrium attaining payoffs within distance ϵ of $(a - l)\frac{\chi}{\phi + \chi}(1, 1)'$.*

This result shows that a separating rule (resulting in reactive signaling dynamics) is optimal whenever the game is played frequently, and that the incentive costs are modest. Intuitively, when the game is played frequently, the costs of signaling a change of type is small (it is incurred once) compared to the benefit of perfectly revealing information (which results in almost perfect information for several rounds of interaction).[26] This implies that as interactions become more frequent, it becomes more likely that players can attain the full benefits of cooperation without incurring significant signaling costs. Indeed, as shown in Section 2, if

[26]The costs of signaling are $\mathcal{O}(\mathcal{D})$ whereas the benefits are $\mathcal{O}(1)$.

players can communicate average total payoffs equal $2(a-l)\frac{1-\mu}{2-\lambda-\mu}$ which, as $D \to 0$, converges to $2(a-l)\frac{\chi}{\phi+\chi}$ –the payoff attained in the game with frequent moves.

This proposition, together with Section 2, show that the effects of reducing the interest rate toward zero are different from those of making the interactions more frequent. When $r \to 0$, the dynamics of cooperation can be time-off, whereas when $D \to 0$ the dynamics of cooperation are reactive signaling. This finding resonates well with those for games with complete information but imperfect monitoring (Abreu, Milgrom, and Pearce 1991, Sannikov and Skrzypacz 2007). While a related point is made in a mechanism design problem by Skrzypacz and Toikka (forthcoming), we seem to be the first ones explicitly studying the differences between low interest rates and frequent interactions in a repeated game model of incomplete information.[27]

7 Conclusions

Oftentimes, economic agents in a long-run relationship can only partially know the conditions under which their partners are making decisions. Moreover, communicating tough or favorable conditions is difficult either because such protocols are non-existent or incomplete (Schelling 1960, Marschak and Radner 1972), or because those conditions materialize only after some other player has already made a decision. We explore the design of optimal equilibria in this type of environment, and show that the dynamics of cooperation are quite rich and novel, and shed light on phenomena that were previously unexplained.

Our results help explain some of the dynamics commonly observed in cooperative relations. First, we explain why economic agents sometimes find it optimal to forgive hostile or aggressive conducts from other agents in a long-run relationship. Second, we explain why to forgive is not to forget: most agents have a limit to the number of aggressions they are willing to tolerate, and the cooperative relationship may end if that limit is surpassed. Third, we show that restarting cooperation after an aggressive conduct has been observed may require costly gestures from the infringing party, or that agents may have to spend a cooling-off period until cooperation is sufficiently likely to be successful again. Our model shows that these behaviors may arise as an efficient way of transmitting information about the likelihood of successful cooperation. Finally, we show that incomplete information may have a significant effect on welfare, even when players are very patient (i.e., have a large valuation for future payoffs), and that in some cases, it may even preclude a cooperative relationship (i.e., cause the optimal equilibrium to consist of constant repetitions of static Nash equilibria).

Our model also explains why firms sometimes engage in unilateral price changes, and gives theoretical support to Stigler's (1947) observation that price leadership may arise optimally as a way to transmit information between competitors. Our model also explains the rationale

[27]In Skrzypacz and Toikka's (forthcoming) model when trade is more frequent, the increase in the persistence of the process of types is detrimental for incentives. In our model, the increase in the persistence of the process helps as signaling a type has more benefits.

behind the commonly observed practice of graduated sanctions and restitutions in common-pool resource settings (Ostrom 1990).

Some extensions to our model would be relatively simple to execute. We have worked with a one-sided incomplete information game to emphasize the forces in the model, but extending the results to allow for two-sided incomplete information entails no challenge.[28] We could also extend our results to allow for restricted communication or communication only once the stage game has been played (but before the subsequent type is realized). It would also be interesting to explore the equilibrium set when the discount factor is not arbitrarily close to 1.[29] The restriction to rules determining unique recurrence classes or pooling paths of actions (Definitions 1 and 2) in Theorem 2 is harder to relax. These conditions are not met when types are not revealed but some types still pool. Countable belief paths in which all probabilities are visited only once but yet the informed player imperfectly reveals could arise.[30] Testing "appropriate behavior" therefore becomes hard under these conditions.[31] The same issue arises when we add imperfect public monitoring to our incomplete-information game (Fudenberg, Levine, and Maskin 1994). Dealing with these extensions is left for future research.

[28] Extending the dynamic programming formulation to a two-sided incomplete information model is immediate. Escobar and Toikka (2013) and Renou and Tomala (2013) design protocols to handle incentives in dynamic models of two-sided incomplete information. Those arguments can be readily applied to our setup under ergodicity restrictions.

[29] Our dynamic programming characterization (4.3) still provides an upper bound for the equilibrium set. Frankel (2015) studies "discounted quotas" and shows their optimality properties in mechanism design problem with iid types and arbitrary patience.

[30] As we have shown, these paths cannot arise when the informed player has only two types, or when the game has strictly monotonic dynamics (Section 5).

[31] Note that lumping beliefs into finitely many subsets is difficult to put to work as the Markov property is lost.

A Appendix

A.1 Proofs and Details for Section 2

Proof of Lemma 1. Convexity and continuity follow since w is the maximum of convex and continuous functions. To see that w is nondecreasing, we prove that whenever w is nondecreasing, so is the function

$$\max\{w_{III}(p), w_{NNN}(p), w_{INI}(p), w_{INN}(p)\}.$$

Observe first that w_{III}, w_{NNN} and w_{INI} are nondecreasing if w is nondecreasing. Write

$$w_{INN}(p) = \delta\, w(1-\mu) + p\big((1-\delta)\,(2b-l) + \delta\, w(\lambda) - \delta\, w(1-\mu)\big)$$

and note that this function is nondecreasing whenever $(1-\delta)\,(2b-l) + \delta\, w(\lambda) - \delta\, w(1-\mu)$ is nonnegative. But if this last term is negative, then

$$w_{INN}(p) \leq \delta\, w(1-\mu) \leq \delta\, w(p\lambda + (1-p)(1-\mu)) = w_{NNN}(p),$$

and therefore the maximization cannot be attained by the rule INN. It follows that $\max\{w_{III}(p), w_{NNN}(p), w_{INI}(p), w_{INN}(p)\}$ is nondecreasing. □

Proof of Lemma 2. We begin by showing there exist $p_1 \in \left(0, \frac{1}{2}\right)$, $p_2 \in (0,1)$, and $p_3 \in [0,1)$ such that: (a) $w_{INI} < w_{INN}$ for $p < p_1$ and $w_{INI} > w_{INN}$ for $p > p_1$, (b) $w_{INI} < w_{NNN}$ for $p < p_2$ and $w_{INI} > w_{NNN}$ for $p > p_2$, and (c) $w_{NNN} < w_{INN}$ for $p < p_3$ and $w_{NNN} > w_{INN}$ for $p > p_3$. We will show how to obtain result (b). The other results are obtained in a similar way.

Recall that:

$$\begin{aligned} w_{INI}(p) &= p\,((1-\delta)\,2(a-l) + \delta\, w(\lambda)) + (1-p)\,((1-\delta)\,(2b-l) + \delta\, w(1-\mu)), \\ w_{NNN}(p) &= \delta\, w(p\,\lambda + (1-p)(1-\mu)). \end{aligned}$$

It is straightforward to see that $w_{INI}(0) < w_{NNN}(0)$ and $w_{INI}(1) > w_{NNN}(1)$. Also, $w_{INI}(p)$ is linear and $w_{NNN}(p)$ is convex (given that $w(p)$ is convex). Thus, the two functions intersect exactly once. Thus, there exists $p_2 \in (0,1)$ such that $w_{INI} > w_{NNN}$ for $p > p_2$ and $w_{INI} < w_{NNN}$ for $p < p_2$.

If joint investment is ever to take place, then $w_{INI}(\lambda) \geq \max\{w_{NNN}(\lambda), w_{INN}(\lambda)\}$. A sufficient condition for $w_{INI}(\lambda) > w_{INN}(\lambda)$ is $\lambda \geq \frac{1}{2}$. By convexity,

$$w(p\,\lambda + (1-p)(1-\mu)) \quad \leq \quad p\, w(\lambda) + (1-p)V(1-\mu).$$

Thus, if $\lambda\, 2\,(a-l) + (1-\lambda)\,(2b-l) > 0$, then $w_{INI}(p) > w_{NNN}(p)$. A sufficient condition is $\lambda \geq \frac{l-2b}{2(a-b)-l}$. Thus, if $\lambda \geq \max\left\{\frac{1}{2}, \frac{l-2b}{2(a-b)-l}\right\}$, the optimal rule dictates play of INI when beliefs are $p = \lambda$.

Now suppose that current play is at INI, and player 1 plays N. Next period, beliefs change to $p = 1 - \mu$. Optimal play at $p = 1 - \mu$ depends on the comparison between w_{INI}, w_{INN}, and w_{NNN}. There are three cases.

First, if $p_1 < 1 - \mu$ and $p_2 < 1 - \mu$, then the optimal play at $p = 1 - \mu$ is INI and the optimal rule is to play INI for any belief p. This is a particular case of time off in which $\hat{\tau} = 0$.

Second, if $1 - \mu < p_2$ and $1 - \mu < p_3$, then the optimal play at $p = 1 - \mu$ is INN. In this case, player 1 plays I if her type is l and plays N if her type is h. Next period, beliefs change to either λ or $1 - \mu$, and optimal play is either INI or INN. Thus, the optimal rule is reactive signaling.

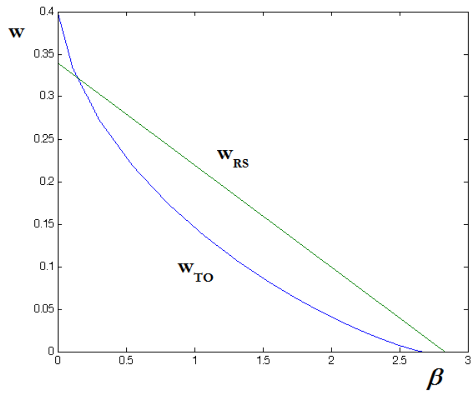
Third, if $p_3 < 1 - \mu < p_2$, then the optimal play at $p = 1 - \mu$ is NNN. Player 1's action does not reveal her type. Next period, beliefs are updated to $p' = (1 - \mu)\,\lambda + \mu\,(1 - \lambda)$. If $p' < p_2$, play continues at NNN. Play continues at NNN until $\mathrm{P}(\theta^t = l) > p_2$, in which case play switches to INI. Thus, the optimal rule is time off with $\hat{\tau} > 0$. Note that if $p_2 > \frac{1-\mu}{2-\lambda-\mu}$, then play will not switch back to INI and the optimal play will be NNN forever. A sufficient condition for $p_2 < \frac{1-\mu}{2-\lambda-\mu}$ is $\frac{1-\mu}{1-\lambda} > \frac{l-2b}{2(a-l)}$. $\square$

We now extend Lemma 2 to provide a complete characterization of optimal dynamics in the undiscounted case. Let $\beta_{TO} = \frac{1-\mu}{(1-\lambda)\,(2-\lambda-\mu)}$ and $\beta_{RS} = \frac{\lambda}{2(1-\lambda)}$.

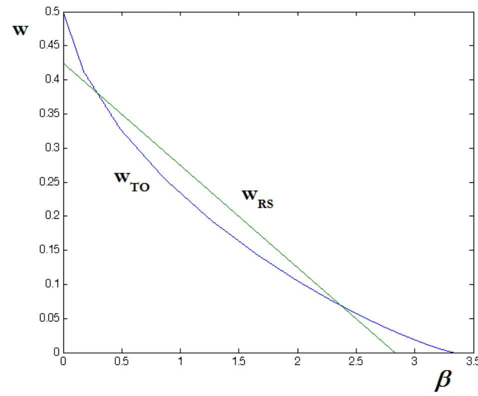
Lemma 6. *Suppose $\delta \to 1$. Reactive signaling leads to positive welfare if and only if $\beta < \beta_{RS}$, and time off leads to positive welfare if and only if $\beta < \beta_{TO}$. Optimal cooperation dynamics depend on parameters as follows: (1) if $\beta_{TO} \leq \beta_{RS}$, there exists a threshold $\beta_0 < \beta_{TO}$ such that time off is better than reactive signaling for $\beta < \beta_0$ and reactive signaling is better than time off for $\beta_0 < \beta < \beta_{RS}$, and (2) if $\beta_{TO} > \beta_{RS}$, then either (a) there exist thresholds β_1 and β_2, with $\beta_1 < \beta_2 < \beta_{RS}$, such that time off is better than reactive signaling for $\beta < \beta_1$ and $\beta_2 < \beta < \beta_{TO}$, and reactive signaling is better than time off for $\beta_1 < \beta < \beta_2$, or (b) time off is better than reactive signaling for all $\beta < \beta_{TO}$.*

Lemma 6 shows there exist three different cases for optimal dynamics as a function of β. Figures 3a, 3b, and 3c present examples of the three cases for different parameter combinations.

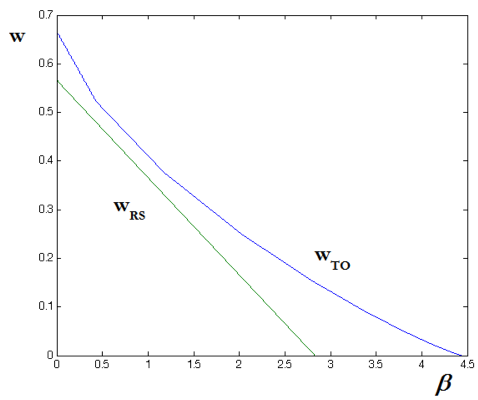
Lemma 6 and Figures 3a, 3b, and 3c show that time off leads to higher welfare than reactive signaling in two cases. First, if the cost of miscoordination is small relative to the opportunity cost of missed cooperation (β is small), then the optimal waiting period τ^* is small. In this case, it is optimal to try to coordinate in joint investment as soon as possible, instead of waiting until player 1 signals her type. Second, if the cost of miscoordination is large relative to the opportunity cost of missed cooperation (β is large) and $\beta_{TO} > \beta_{RS}$, players want to avoid the cost of miscoordination if possible. If the stationary probability of



(a) $\lambda = 0.85$, $\mu = 0.9$



(b) $\lambda = 0.85$, $\mu = 0.85$



(c) $\lambda = 0.85$, $\mu = 0.7$

Figure 3: Welfare comparisons

$\theta = l$ is large enough, players can avoid this cost by waiting for a long time before trying to cooperate again. In formal terms, if $\beta_{TO} > \beta_{RS}$ and β is close to β_{TO}, then the optimal waiting period τ^* is large ($\tau^* \to \infty$ as $\beta \to \beta_{TO}$) but time off is better than reactive signaling.

Proof of Lemma 6. At time 0, the expected values of the RS and TO rules when $\delta \to 1$ are:

$$\begin{aligned} w_{RS} &= \frac{(1-\mu)\,(\lambda - 2\,\beta\,(1-\lambda))}{2-\lambda-\mu}, \\ w_{TO} &= \frac{P(\tau) - \beta\,(1-\lambda)}{(\tau+1)(1-\lambda) + P(\tau)}, \end{aligned}$$

where

$$P(\tau) = \frac{(1-\mu)\left(1 - (\lambda+\mu-1)^{\tau+1}\right)}{2-\lambda-\mu}$$

is the probability that $\theta^{t+\tau+1} = l$, given that $\theta^t = h$. See Appendix B for details on how to obtain these expressions.

We begin by showing that $\hat{\tau}$ (the optimal length of the waiting phase in TO) is nondecreasing in β. The second derivative of w_{TO} with respect to T and β is

$$\frac{\partial^2 w_{TO}}{\partial\tau\,\partial\beta} = \frac{(1-\lambda)\,\big((1-\lambda)+\big(1-P(\tau)\big)\,\log(\lambda+\mu-1)\big)}{\big((\tau+1)(1-\lambda)+P(\tau)\big)^2},$$

which is positive. Therefore, $\hat{\tau}$ is nondecreasing in β.

We now show that value decreases with β for both RS and TO. The derivatives of w_{RS} and w_{TO} with respect to β are:

$$\begin{aligned}\frac{\partial w_{RS}}{\partial\beta} &= -\frac{2\,(1-\lambda)\,(1-\mu)}{2-\lambda-\mu},\\ \frac{w_{TO}}{\partial\beta} &= -\frac{1-\lambda}{(\tau+1)(1-\lambda)+P(\tau)}.\end{aligned}$$

Notice also that the derivative of w_{RS} is constant with respect to β, but the derivative of w_{TO} is nondecreasing in β (that is, it decreases in absolute value as β increases), because $\hat{\tau}$ is nondecreasing in β. This means that w_{RS} is linear and w_{TO} is convex with respect to β.

It is easy to see that for β sufficiently large, both w_{RS} and w_{TO} are negative, thus RS and TO are dominated by a pooling rule in which players never invest. In particular, $w_{RS} > 0$ iff $\beta < \beta_{RS}$ and $w_{TO} > 0$ iff $\beta < \beta_{TO}$, where $\beta_{RS} = \frac{\lambda}{2\,(1-\lambda)}$, and $\beta_{TO} = \frac{1-\mu}{(1-\lambda)\,(2-\lambda-\mu)}$.

Now we proceed to compare RS and TO. First, note that for RS to be optimal, $w_{RS} \geq w_{TO}$ when $T = 0$ (if RS is worse than TO when $T = 0$, then it is also worse at the optimal $\hat{\tau}$). This implies that a necessary condition for RS to be optimal is that $\beta \geq \frac{1-\mu}{2\,\mu-1}$. Thus, TO dominates RS for small β.

Suppose that $\beta_{TO} < \beta_{RS}$. Given continuity of w_{TO} and w_{RS} with respect to β, and the convexity of w_{TO}, there exists exactly one point in which the two lines cross. Thus, there exists a threshold $\beta_0 < \beta_{TO}$ such that TO dominates RS for $\beta < \beta_0$ and RS dominates TO for $\beta_0 < \beta < \beta_{RS}$.

Finally, suppose that $\beta_{TO} > \beta_{RS}$. Given continuity of w_{TO} and w_{RS} with respect to β, and the convexity of w_{TO}, we know that close to β_{TO} (i.e., for large β) TO must dominate RS. There are two cases: (a) w_{RS} may be above w_{TO} for intermediate values of β, or (b) w_{RS} may lie below w_{TO} for all β. $\square$

A.2 Proofs for Section4.1

Proof for Lemma 4. The result is the standard dynamic programming formulation of partially observed Markov decision processes (Arapostathis, Borkar, Fernández-Gaucherand, Ghosh, and Marcus 1993). A minor subtlety arises due to the fact that our control variables are mixed strategies which, in contrast to what is typically addressed in the literature, involve private randomizations. To address this, note that a decision rule can be equivalently written as $f = (f_i^t)$ with $f_i^t\colon A^{t-1} \times \Theta_i^t \times [0,1]^t \times [0,1] \to A_i$, where the last component of an element in the range only determines the action of player i. In other words, $a_i^t = f_i^t(a^1, \ldots, a^{t-1}, \theta_i^1, \ldots, \theta_i^t, \chi^1, \ldots, \chi^t, \chi_i^t)$ where χ_i^t is only used by i. We can expand the set over which the maximization (4.1) is performed by allowing rules where all players at t condition on the whole vector $(\chi_1^t, \ldots, \chi_N^t)$. This relaxed efficiency problem admits a dynamic programming formulation in which, without loss, public randomizations are not used. Since the solution of the relaxed problem is feasible for (4.1), we deduce that $q(\alpha) = w^{\alpha,\delta}(\lambda)$.

□

Proof of Theorem 1. We use the so-called vanishing discount approach. Parts a. and b. follow from Platzman (1980) or Theorem 11 in Hsu, Chuang, and Arapostathis (2006). It is enough to note that the hidden Markov process $(\theta^t)_{t\geq 1}$ has full support and note that, for example, Assumption 2 in Hsu, Chuang, and Arapostathis (2006) holds. To deduce c, we use part (d) Corollay on p.369 in Platzman (1980).

□

Proof of Proposition 1. Consider the problem

$$\max_{\sigma\in\Sigma} \sum_{a_1\in A_1} h(B(\cdot \mid \sigma_1, p, a_1)) \sum_{\theta\in\Theta, \sigma_1(\theta)=a_1} p(\theta)$$

with $h\colon \Delta(\Theta) \to \mathbb{R}$ convex. The solution is any separating rule (in particular, $\bar{\sigma}(\cdot \mid \bar{p})$ in the text solves this problem). To see this, notice that the problem can be reformulated as the problem of choosing a Bayes-consistent belief distribution over beliefs with the purpose of maximizing a convex function (Gentzkow and Kamenica 2011). The value of that problem equals the concave hull of the objective and is attained by a distribution putting appropriate weights over delta-Dirac beliefs.

□

A.3 Proofs for Section 4.2

Proof of Lemma 5. Let us first prove a. Since the control rule σ^α determines an irreducible Markov chain (Definition 1), there exists an irreducible transition matrix $\bar{P}$ for the joint process of states and beliefs, $(\theta^t, p^t)_{t\geq 1} \in \Theta \times P$ and a unique stationary distribution $\bar{\pi}$ on $\Theta \times P$. Using Blackwell's (1957) construction, we can extend the Markov chain (θ^t, a^t) to the negative numbers $t \in \mathbb{Z}$, and compute the invariant measure $\bar{\pi}(\theta, p) = \mathrm{P}\{\theta_0 = \theta, \mathrm{P}[\theta_0 = \cdot \mid (a^t)_{t\leq 0}] = p(\cdot)\}$. In particular, for any $(\theta, p) \in \Theta \times P$,

$$\bar{\pi}(\theta \mid p) = \mathrm{P}\Big[\theta^0 = \theta \mid p = \mathrm{P}[\theta^0 = \cdot \mid (a^t)_{t\leq 0}]\Big] = p(\theta). \tag{A.1}$$

Now, for any sequence $(\theta_t, p_t)_{t\geq 1}$, we define the empirical transition matrix $\bar{P}^t$ on $\Theta \times P$ as

$$\bar{P}^t\big((\theta', p') \mid (\theta, p)\big) = \frac{|\{t' \leq t-1 \mid (\theta_{t'}, p_{t'}) = (\theta, p), (\theta_{t'+1}, p_{t'+1}) = (\theta', p')\}|}{|\{t' \leq t-1 \mid (\theta_{t'}, p_{t'}) = (\theta, p)\}|}.$$

and the empirical measures

$$\bar{\pi}^t(\theta, p) = \frac{1}{t} \sum_{t'=1}^{t} \mathbb{1}_{(\theta^{t'}, p^{t'})=(\theta,p)}$$

and

$$\bar{\pi}^t(p) = \sum_{\theta\in\Theta} \bar{\pi}^t(\theta, p) = \frac{1}{t} \sum_{t'=1}^{t} \mathbb{1}_{p^t=p}.$$

Finally, define

$$N^t(\theta, p) = \sum_{t'=1}^{t} \mathbb{1}_{(\theta^{t'}, p^{t'})=(\theta,p)}$$

for $(\theta, p) \in \Theta \times P$.

Our first observation is that there exists a constant $c_1 > 0$ (depending on $\bar{P}$ and $\bar{\pi}$) such that for any $t \geq 1$ and an empirical transition matrix $\bar{P}^t$ on $\Theta \times P$ sufficiently close to $\bar{P}$,

$$\|\bar{\pi}^t - \bar{\pi}\| \leq c_1 \|\bar{P}^t - \bar{P}\| + c_1 \frac{1}{t}$$

where $\|\cdot\|$ is the supreme norm. To see this inequality, we borrow the following two formulas from Lemma B.2 in Escobar and Toikka (2013)

$$\bar{\pi}^t = \Big(I - \bar{P}^t + E\Big)^{-1} (\mathbb{1} + e^t), \quad \bar{\pi} = \Big(I - \bar{P} + E\Big)^{-1} \mathbb{1}$$

where $\|e^t\| \leq \frac{|\Theta||P|}{t}$ and note that the map $\bar{P}' \mapsto \Big(I - \bar{P}' + E\Big)^{-1}$ is Lipschitz in a neighborhood of $\bar{P}$. Moreover, since $\bar{\pi}(\theta, p) > 0$ for all $\theta \in \Theta$ and all $p \in P$, without loss we can take c_1 such that

$$\|\frac{\bar{\pi}^t(\theta, p)}{\bar{\pi}^t(p)} - \bar{\pi}(\theta \mid p)\| \leq c_1 \|\bar{P}^t - \bar{P}\| + c_1 \frac{1}{t}$$

for all $(\theta, p) \in \Theta \times P$. Combining this observation with (A.1) we deduce that for all $p \in P$

$$\|\bar{\pi}^t(\cdot \mid p) - p(\cdot)\| \leq c_1 \|\bar{P}^t - \bar{P}\| + c_1 \frac{1}{t} \tag{A.2}$$

Now, ignore the moves of player 0 and assume that player 1's actions are never modified. Use Lemma B.1 in Escobar and Toikka (2013) to show that there exists a decreasing sequence $(d_k)_k$ converging to 0 such that

$$\mathrm{P}_{\hat{s}_1}[\|\bar{P}^t(\cdot \mid (p, \theta)) - \bar{P}(\cdot \mid (p, \theta))\| < d_{N^t(p,\theta)} \quad \forall t \geq 1, \forall (\theta, p)] \geq 1 - \frac{\epsilon}{2}. \tag{A.3}$$

Fix $0 < \psi < \min_{\theta,p} \bar{\pi}(\theta, p)$ and use Theorem 1.10.2 in Norris (1997) to find $\bar{t}$ such that

$$\mathrm{P}_{\hat{s}_1}[N^t(p, \theta) \geq t(\bar{\pi}(\theta, p) - \psi), \ \forall t \geq \bar{t}] \geq 1 - \frac{\epsilon}{2}. \tag{A.4}$$

Define $c_2 = \min_{\theta,p} \bar{\pi}(\theta, p)(> 0)$ and the sequence $(b_k)_k$ by $b_k = c_1|\Theta|\big(d_{k(c_2-\psi)} + \frac{1}{k}\big)$ for all $k \geq \bar{t}$ (for $k < \bar{t}$, $b_k = 2$). From (A.2), (A.3), and (A.4)

$$\mathrm{P}_{\hat{s}^1}[\|\bar{\pi}^t(\cdot \mid p) - p(\cdot)\| \leq \frac{1}{|\Theta|} b_t \quad \forall t \geq 1, p \in P] \geq 1 - \epsilon.$$

Note that for any element of the event above, player 1 passes the test (b_k) because

$$\max_{a_1 \in A_1} \|m^t(a_1 \mid p) - m(a_1 \mid p)\| \leq |\Theta| \|\bar{\pi}^t(\cdot \mid p) - \bar{\pi}(\cdot \mid p)\| \leq b_t$$

and therefore

$$\mathrm{P}[1 \text{ passes test } (b_k) \text{ at } (a^1, \ldots, a^t),\ \forall t] \geq 1 - \epsilon.$$

It follows that we introduce the possibility that player 0 changes player 1's actions after failing a test, the lower bound for the probability above remains unaltered.

We now prove b. Construct the test (b_k) from part a given ϵ. Observe that player 1 can always use the obedient strategy $\hat{s}^1$ in a game of credible play, and therefore his average payoff is at least $v_1^\alpha - \epsilon$. Now, there exists $\bar{T} \geq 1$ such that for any $T \geq \bar{T}$, and any strategy s_1 for player 1 in the credible reporting game $(\sigma^\alpha, (b_k), T)$,

$$\mathrm{P}_{s_1}[\|m^T(\cdot \mid p) - m(\cdot \mid p)\| \leq \epsilon, \forall p \in P] \geq 1 - \epsilon.$$

This observation follows by noticing that regardless of the strategy s_1 used by 1, if at any given round player 1 fails the test, the continuation actions are drawn from $m(\cdot \mid p)$ (see Lemma B.5 in Escobar and Toikka (2013)). Therefore, with sufficiently high probability, for any strategy s^1, Player 1 passes a relaxed test at the end of the block given the history of actions $(a^1, \ldots, a^T)$. In particular, the expected average payoff for player 2 over the course of a game of credible play is within ϵ of v_2^α. It therefore follows that for all $\epsilon > 0$, for any best response s^1 for player 1 in the game of credible play $(\sigma^\alpha, (b_k), T)$, with $T \geq \bar{T}$, Player 1 payoff is at least $v_1^\alpha - \epsilon$ and player 2's payoff is within ϵ of v_2^α. Introducing discounting and putting the games of credible play back-to-back to form a block-game of credible play, we deduce that for any best response $s_1 = s_1^\delta$ of player 1 in $(\sigma^\alpha, (b_k), T)^\infty$, for $\delta \geq \delta(T)$,

$$v_1^\delta(s_1) \geq v_1^\alpha - \epsilon$$

and $\|v_2^\delta(s_1) - v_2^\alpha\| \leq \epsilon$. Now, extend Corollary 1 in Platzman (1980) to deduce that[32]

$$\limsup_{\delta \to 1} \alpha \cdot v^\delta(s^\delta) \leq \rho^\alpha.$$

We deduce that $\lim_{\delta \to 1} \alpha \cdot v^\delta = \rho^\alpha = \alpha \cdot v^\alpha$. The result follows. □

[32]The extension is due to the fact that we need convergence uniformly across strategies, but the very same proof by Platzman (1980) works in our setup.

Proof of Theorem 2. Equilibrium strategies can be described as follows. Players start in a *cooperative phase* by choosing actions as in the equilibrium of the games of credible play $(\sigma^{\alpha}, (b_k), T)^{\infty}$. Any observable deviation by player i triggers a *stick phase* in which the players play minmax against i during L periods. Any deviation by a player restart a minmax phase of L rounds against that player. After the L rounds of minmax against i, a *carrot phase* is started in which players choose actions as in the equilibrium of the game of credible play $(\sigma^{\alpha^i}, (b_k), T)^{\infty}$. Deviations restart the minmax phase and so on.

Let $\epsilon > 0$ be small enough such that for some $\gamma \in]0, 1[$

$$v_i^{\alpha^{-i}} - v_i^{\alpha^i} > 2\epsilon, \quad (1-\gamma) > \frac{2\epsilon}{v_i^{\alpha^i} - \underline{v}_i}, \quad \gamma\Big(v_i^{\alpha^{-i}} - v_i^{\alpha^i} - 2\epsilon\Big) > (1-\gamma)\Big(\underline{v}_i - m + \epsilon\Big)$$

for $i = 1, 2$. Take $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ the credible reporting games $(\sigma^{\alpha'}, (b_k), T)^{\infty}$, for $\alpha' = \alpha, \alpha^1, \alpha^2$, have discounted equilibrium payoffs $U^{\alpha'}(\delta)$ within distance ϵ of the target payoffs $v^{\alpha'}$. Define the length of the stick phase as $L(\delta) = \max\{d \in \mathbb{N} \mid d \leq \frac{\ln(\gamma)}{\ln(\delta)}\}$ and note that $\delta^L \to \gamma$. Lemma 6.1 in Escobar and Toikka (2013) shows that discounted payoffs during the L periods of the stick phase against i are bounded above by $(1-\delta^L)(\underline{v}_i + \epsilon)$ for δ sufficiently large.

Now, consider the incentives in the carrot phase

$$v_i^{\alpha} - \epsilon \geq (1-\delta)M + (\delta - \delta^{L+1})(\underline{v}_i + \epsilon) + \delta^{L+1}(v_i^{\alpha^i} + \epsilon)$$

The incentives of player i in the stick phase against $j \neq i$ can be written

$$(1-\delta^L)m + \delta^L(v_i^{\alpha^j} - \epsilon) \geq (1-\delta)M + (\delta - \delta^{L+1})(\underline{v}_i + \epsilon) + \delta^{L+1}(v_i^{\alpha^i} + \epsilon)$$

Finally, the incentives of player i in the carrot phase against j can be written as

$$v_i^{\alpha^j} - \epsilon \geq (1-\delta)M + (\delta - \delta^{L+1})(\underline{v}_i + \epsilon) + \delta^{L+1}(v_i^{\alpha^i} + \epsilon)$$

Taking the limit as $\delta \to 1$ in all these inequalities, by construction of ϵ and γ, we deduce the existence of a critical discount factor such that all incentive constraints hold. □

A.4 Proofs for Section 5

Proof of Proposition 3. Consider first a solution $\sigma^* \in \Sigma$ to the problem

$$\max_{\sigma \in \Sigma} \sum_{\theta \in \Theta} \Big(\alpha_1 u_1(\sigma_1(\theta), \sigma_2, \theta) + \alpha_2 u_2(\sigma_1(\theta), \sigma_2)\Big) p(\theta)$$

Since $p(\theta) > 0$ for all θ,

$$\sigma_1^*(\theta) \in \arg\max_{a_1 \in A_1} \Big\{\alpha_1 u_1(a_1, \sigma_2^*, \theta) + \alpha_2 u_2(a_1, \sigma_2^*)\Big\}.$$

Fix $\theta^m \in \Theta$ with $m < |\Theta|$ and $a^n = \sigma_1^*(\theta)$ with $2 \leq n \leq |A_1| - 1$. By concavity, the derivative

$$\frac{\partial}{\partial a_1}\Big(\alpha_1 u_1 + \alpha_2 u_2\Big)(a^{n-1}, \sigma_2^*, \theta^m)$$

is nonnegative. Now,

$$\begin{aligned}\frac{\partial}{\partial a_1}\Big(\alpha_1 u_1 + \alpha_2 u_2\Big)(a^{n+1}, \sigma_2^*, \theta^{m+1}) &= \frac{\partial}{\partial a_1}\Big(\alpha_1 u_1 + \alpha_2 u_2\Big)(a^{n-1}, \sigma_2^*, \theta^m) \\ &+ \int_{a^{n-1}}^{a^{n+1}} \frac{\partial^2}{\partial a_1^2}\Big(\alpha_1 u_1 + \alpha_2 u_2\Big)(y, \sigma_2^*, \theta^m)dy + \alpha_1 \int_{\theta^m}^{\theta^{m+1}} \frac{\partial^2}{\partial a_1 \partial \theta} u_1(a^{n-1}, \sigma_2^*, y)dy \\ &\geq |a^{n+1} - a^n|(-c_1) + \alpha_1 c_2(\theta^{m+1} - \theta^m)\end{aligned}$$

is positive under (5.3). It follows that $\sigma_1^*(\theta^{m+1} \mid p) \geq a^{n+1} > \sigma_1^*(\theta^m \mid p)$.

To deduce the second part of the Proposition, use the results in Van Zandt and Vives (2007) for monotone comparative statics in Bayesian games.

□

A.5 A Proof for Section 6

Proof of Proposition 4. Lemma 2 shows that the optimal equilibrium follows either reactive-signaling or time-off dynamics. The limit of the value of playing reactive-signaling when $D \to 0$ is

$$\lim_{D \to 0} w_{RS} = 2\,(a - l)\,\frac{\chi}{\phi + \chi}.$$

The limit of the value of playing time-off for a given τ when $D \to 0$ is

$$\lim_{D \to 0} w_{TO}(\tau) = \frac{\chi\,(\tau + 1)\,2\,(a - l) + \phi\,(2b - l)}{(\phi + \chi)\,(\tau + 1)},$$

and the derivative of this expression with respect to τ is

$$-\frac{\phi\,(2b - l)}{(\phi + \chi)\,(\tau + 1)^2} > 0,$$

Thus, as $D \to 0$, $\hat{\tau} \to \infty$, and limit of the value of playing time-off when $D \to 0$ is

$$\lim_{D \to 0} w_{TO} = 2\,(a - l)\,\frac{\chi}{\phi + \chi},$$

which is equal to the limit value of playing reactive signaling. This result is very intuitive. As $D \to 0$, the process of types becomes perfectly persistent, and the probability of a type change is equal to 0. In the first period of play, the probability that player 1 has low cost is $\chi/(\phi + \chi)$. Thus, the value of playing either reactive signaling or time off is $2\,(a - l)\,\frac{\chi}{\phi + \chi}$.

In order to compare the two rules, we compare the derivatives of the limit value with respect to D, as $D \to 0$ from the right. For reactive signaling, we have

$$\lim_{D\to 0} \frac{\partial w_{RS}}{\partial D} = \left(-\chi\,(a-l) + (2\chi + r)\,(2b-l)\right) \frac{\chi}{\phi+\chi},$$

and for time off, we have

$$\lim_{D\to 0} \frac{\partial w_{TO}}{\partial D} = -\infty.$$

Both derivatives are negative, but the derivative corresponding to time off is larger in absolute value. Thus, to the left of $D = 0$, reactive signaling has greater value than time off. This proves part a of the proposition.

To prove b, we follow steps close to those in the proof of Theorem 2. The definition of game of credible reporting remains unaltered for any given D. We will prove that for a proper choice of parameters, we can replicate Lemma 5. We construct the sequence b_k from the definition of d_k (see proof of Lemma 5) by picking $0 < \psi < \lim_{D\to 0} \bar{\pi}^D(\theta, p)$, with $\bar{\pi}^D$ the stationary distribution given D, and $b_k = c_1|\Theta|(d_{k(c_2-\psi)} + \frac{1}{k})$. Conditions (A.2) and (A.3) follow immediately for any D. Condition (A.4) is also immediate, just notice that the choice of $\bar{t}$ depends on D so $\bar{t} = \bar{t}(D)$. This completes the first part of Lemma 5. To see the second part, construct $\bar{T} = \bar{T}(D)(> \bar{t}(D))$ so that for any strategy s_1 $\mathbb{P}^D_{s_1}[\|m^T(\cdot \mid p) - m^D(\cdot \mid p)\| \leq \epsilon \quad \forall p \in P^D] \geq 1 - \epsilon$. Note that for the game of credible play $(\bar{\sigma}, (b^D_k), T)$, with $T \geq \bar{T}(D)$, Player 1 can obtain a payoff at least $(a-l)\frac{\chi}{\phi+\chi} - \epsilon$. By construction, Player 2's payoff is within ϵ of $(a-l)\frac{\chi}{\phi+\chi}$. Fixing τ, $T \geq \bar{T}(D)$, we can find $\bar{r}(D)$ such that for all $r < \bar{r}(D)$, for any best response s_1 in the block-game of credible play, Player 1 obtains a payoff at least $(a-l)\frac{\chi}{\phi+\chi} - \epsilon$. Taking $D \leq \bar{D}$ and $r \leq \bar{r}(D)$ (sufficiently small if needed), by definition equilibrium payoffs in the game played every D units of time with discount rate r are bounded above by $2(a-l)\frac{\chi}{\phi+\chi} + \epsilon$. Observable deviations from the path of play of the block-credible reporting game are punished by Nash reversion. Provided $\bar{r}(D)$ is chosen sufficiently small, the result follows. □

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