Coalitional Expected Multi-Utility Theory

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Abstract

This paper begins by observing that any reflexive binary (preference) relation (over risky prospects) which satisfies the Independence Axiom admits a form of expected utility representation. We refer to this representation notion as *coalitional minmax expected utility representation.* By adding the remaining properties of the expected utility theorem, namely, continuity, completeness and transitivity, one by one, we find how this representation gets sharper and sharper, thereby deducing the versions of this classical theorem in which any combination of these properties are dropped from its statement. This approach also allows us to weaken transitivity in this theorem, rather than eliminating it entirely, say, to quasitransitivity or acyclicity. Apart from providing a unified dissection of the expected utility theorem, these results are relevant for the growing literature on boundedly rational choice in which revealed preference relations often lack the properties of completeness and/or transitivity (but often satisfy the Independence Axiom). Finally, and perhaps more importantly, we show that our representation theorems allow us to answer many economic questions that are posed in terms of nontransitive/incomplete preferences, say, about the maximization of preferences, existence of Nash equilibrium, preference for portfolio diversification, and possibility of the preference reversal phenomenon.

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1 Introduction

The *expected utility theorem* is one of the foundational pillars of modern economic theory. This theorem, which goes back to the 1947 contribution of John von Neumann and Oskar Morgenstern, says that preferences of a "rational" individual over risky prospects (lotteries) would be represented by the expectation of a (cardinal) utility function over the payouts of the lotteries. Needless to say, this result has led to the emergence of modern decision theory under risk and uncertainty, and has played a foundational role in many other fields, ranging from game theory to financial economics.

At its core, the expected utility theorem takes a reflexive binary relation on a space of lotteries, and imposes four hypotheses on this relation: Completeness, transitivity, continuity and the Independence Axiom. While the continuity hypothesis is a basic (and empirically untestable) regularity condition, the remaining three postulates are behavioral properties that correspond to various types of "rationality" on the part of the decision maker. As such, numerous experimental studies have shown that they are violated by many individuals, and this has led decision theorists pursue generalizations of the expected utility theorem by weakening some of these hypotheses. Motivated by the Allais' Paradox and its derivatives, most work in this regard has concentrated on weakening the Independence Axiom, thereby leading to what is now known as "non-expected utility theory." However, far less attention is given to what happens to the expected utility theorem when we relax some or all of the remaing three postulates of the theory. A notable exception to this is the work of Aumann (1962), which initiated the research on obtaining an expected utility theorem in which all but the completeness axiom is assumed. (Recently, Dubra, Maccheroni and Ok (2004) have proved that result precisely.) Another exception is Hausner and Wendel (1952), where it was asked and answered exactly what would happen to the expected utility theorem if we dropped the continuity hypothesis from its postulates. Surprisingly, and despite ample empirical evidence of non-transitive choice behavior under risk, a similar exercise was not carried out with respect to the transitivity hypothesis, and/or with respect to combinations of the properties of completeness, transitivity and continuity. The only work related to this issue is a series of papers by Peter Fishburn (starting from 1982) in which he has developed the so-called skew-symmetric bilinear (SSB) utility theory (a precursor of which can be found in Kreweras (1961)). However, this theory allows for nontransitivity only through violations of the Independence Axiom (in the sense that SSB utility that satisfies the Independence Axiom is transitive, and hence reduces to the classical expected utility theory).

All in all, even after 68 years and despite its foundational importance, the literature does not provide a complete analysis of what exactly happens to the expected utility theorem if we drop some or all of the hypotheses of completeness, transitivity, and/or continuity in its statement. In a nutshell, the primary purpose of the present paper is to carry out this exercise. Given the central place the expected utility theorem occupies within decision theory, this appears amply justified. However, our motivation for this goes beyond specialized interest, as we outline next.

First Motivation. Even though its descriptive validity leaves much to be desired, the Independence Axiom is used in a wide variety of decision-theoretic settings that involve risk, ranging from choice under uncertainty and ambiguity to intertemporal choice and menu preferences. Considering also the normative appeal of this postulate, therefore, understanding what this axiom alone entails about one's preferences (which is not assumed to be either complete or transitive at the outset) seems desirable. Put differently, the question is how the expected utility theorem would modify if we dropped all three of the completeness, transitivity and continuity axioms. If we could answer this question, by adding these properties one by one to the system of postulates at hand, we could isolate their individual contributions to the expected utility theorem. At the very least, this would provide a complete picture of the iceberg the tip of which is the von Neumann-Morgenstern expected utility theorem.

Indeed, we find here that any reflexive binary relation (over the set of all lotteries on a separable metric space) which satisfies the Independence Axiom admits quite an interesting representation. This representation, which we refer to as *coalitional* (*minmax*) *expected utility representation*, maintains that there is a set $\mathbb{U}$ of sets (coalitions) of continuous utility functions such that a lottery p is preferred to a lottery q iff for every coalition $\mathcal{U}$ in $\mathbb{U}$ there is a utility function in $\mathcal{U}$ whose expectation with respect to p is at least as large as its expectation with respect to q. If we think of each utility function here as corresponding to a rational "self" of the agent, then we may interpret her preference of p over q as being a consequence of no coalition of her rational "selves" blocking p in favor of q.

The main theoretical finding of the present work is that one can characterize any relaxation of the completeness, transitivity and continuity properties in the expected utility theorem in terms of the notion of coalitional expected utility representation. Adding any one of these properties to the system restricts the structure of $\mathbb{U}$ in a particular way, and brings us closer to the expected utility theorem. In particular, if we add continuity (and assume that the underlying space of riskless prizes is compact), we find that one can choose each of the members of $\mathbb{U}$ as compact collections (as well as $\mathbb{U}$ itself), and in that case we can express our representation more concisely: The agent prefers p over q iff

$$\min\left\{\max_{u\in\mathcal{U}}(\mathbf{E}(u,p)-\mathbf{E}(u,q)):\mathcal{U}\in\mathbb{U}\right\}\geq 0,$$

where $\mathbf{E}(u,p)$ is the expectation of u with respect to p, and similarly for $\mathbf{E}(u,q)$. If, further, we assume that preferences are transitive, then we can choose each of the members of $\mathbb{U}$ as singletons, and recover the expected multi-utility representation of Dubra, Maccheroni and Ok (2004). If, instead, we assume that preferences are complete, we find that $\mathbb{U}$ must exhibit a particular coherence property, namely, any two elements (coalitions) in $\mathbb{U}$ must have a common agent. This result, from which many other characterizations follow, tells us exactly how the expected utility theorem reads without the transitivity axiom. As such, it parallels the work of Dubra, Maccheroni and Ok (2004), but for nontransitive preferences, instead of incomplete ones.

Unlike the property of completeness, there are general ways of weakening the transitivity property, which may be suitable to adopt in certain settings of economic interest. For instance, if one wishes to allow for cyclic choices to arise only due to perception difficulties (the case of intransitive indifference), we may postulate that preferences are quasitransitive (that is, the strict part of the relation is transitive). Or, if we wish to impose that the agent is always able to identify a best alternative within any finite collection of lotteries, we would impose that preferences are acyclic. The notion of coalitional expected utility representation remains operational to deal with either of these cases. In particular, we find here that if preferences are quasitransitive, and satisfy all axioms of the expected utility theorem but transitivity, then we can choose $\mathbb{U}$ in the representation itself as a singleton, thereby obtaining a representation which is very much in the same spirit with what Lehrer and Teper (2011) call *justifiable preferences.* On the other hand, we prove that if preferences are acyclic, and satisfy all axioms of the expected utility theorem but transitivity, then we can choose $\mathbb{U}$ in such a way that *all* elements (coalitions) possess a common element. The applications we consider in the second part of the paper demonstrate that this "strong coherence" property provides a remarkably operational structure for the model.

Second Motivation. In a variety of contexts, one needs to consider as a primitive "preference relation" a binary relation that may be neither complete nor transitive. In particular, since the seminal contributions of Aumann (1962) and Bewley (1986), many authors have argued that completeness is not an unexceptionable trait of rationality. There is now a fairly sizable literature on rational decision making with incomplete preferences in a variety of contexts, ranging from consumption choice to decision making under risk and uncertainty. And there is even a larger literature that works with nontransitive preferences. This literature has mostly a "boundedly rational" flavor, and it studies topics such as nontransitive indifferences that arise from perception difficulties (cf. Luce (1956)), or procedural decision making by using similarity comparisons or regret considerations (cf. Rubinstein (1988) and Loomes and Sugden (1982)). There are important works on the theory of demand and competitive equilibrium (mostly under certainty) in which agents' preferences are allowed to be nontransitive (cf. Shafer (1974) and Kim and Richter (1986)). Starting with Tversky (1969), there are also numerous experimental findings that point to the existence of even strict cycles in the choices of individuals over risky prospects. Besides, when we consider the preference relation at hand as that of a group of individuals (as in social choice theory), it becomes only natural to allow for its lack of transitivity. We refer to Fishburn (1991) and Nishimura (2015) for thorough discussions of the importance of studying nontransitive preference relations from both normative and descriptive perspectives, as well as examples of economic contexts in which nontransitive preferences play an important role.

Third Motivation. In revealed preference theory one arrives at a "preference relation" endogenously, and in many cases of interest it is not possible to write down conditions on choice correspondences that would guarantee either the completeness or the transitivity of this relation. The recent literature on boundedly rational choice theory provides numerous illustrations of this situation.[1] When specialized to the context of risky prospects, however, one can always impose the condition that p is chosen from a set S iff $\lambda p + (1-\lambda)r$ is chosen from the set $\{\lambda q + (1-\lambda)r : q \in S\}$ for any $\lambda \in [0,1]$ and lottery r, and this often entails that the revealed preference relation(s) (or attention sets, etc.) can be chosen to satisfy the Independence Axiom. In such a situation, deepening the coverage of the characterization of choice correspondences requires one to know the structure of those nontransitive and/or

[1] In Eliaz and Ok (2006), for instance, revealed preference relations are incomplete, and in Heller (2012) and Cherepanov, Feddersen and Sandroni (2013), they are nontransitive. On the other hand, Manzini and Mariotti (2007), Masatlioglu and Ok (2014), and Ok, Ortoleva and Riella (2015) use the revealed preference method to obtain what they call "psychological constraint relations" which need not be either complete or transitive.

incomplete binary relations that satisfy the Independence Axiom.[2] The main representation theorems of this paper are primed for this purpose.

Fourth Motivation: *Applications.* Broadly speaking, utility representation theorems have two main uses. First, they provide a functional structure for a preference relation that allows one to model that type of a relation in economic environments. Second, they sometimes help us answer economic questions based on certain types of preferences, which may be quite difficult to deal with by using the defining properties of these preferences directly.

We found that the representation theorems we provide in this paper are particularly useful from this perspective, as exemplified by the large number of applications we present later in the paper. To hint at what we mean by this, let us consider an individual whose preferences satisfy all postulates of the expected utility theorem, but transitivity. Assume further that this individual is not overly nontransitive in that her preferences are acyclic (which, in particular, implies that she would not fall prey to a strict money pump scheme). Consider the following questions:

> (1) Among all lotteries over a finite set of (riskless) prizes, is there a degenerate lottery that is best for this individual?
>
> (2) Suppose this individual plays a strategic game against another such individual. Does there exist a mixed strategy Nash equilibrium of that game?
>
> (3) Suppose the lotteries in question are monetary and the preferences of the agent are consistent with first order stochastic dominance. Is it possible that this individual prefers a lottery p over q strictly, and yet her minimum selling price for p is strictly lower than that of q (the preference reversal phenomenon)? What if her preferences were quasitransitive (so that her cycles may arise only due to intransitive indifference)?
>
> (4) Suppose the lotteries in question are monetary and the preferences of the agent are consistent with second order stochastic dominance (so that she is risk averse). Does this agent exhibit preference for portfolio diversification?

None of the authors of this paper were able to answer these questions (except the second part of (3)) without invoking at least one of the representation theorems we prove in Section 3. With those theorems, however, we were able come up with the answers fairly easily.[3] In Section 4, we use these theorems to ask and answer even more general versions of these questions (and quite a few more).

[2]For instance, Eliaz and Ok (2006) invoke the expected multi-utility theorem of Dubra, Maccheroni and Ok (2004) to provide a functional structure for the revealed (incomplete) preferences in their choice model. To be honest, the present work was initiated when we attempted to carry out the same exercise in the context of the revealed (p)reference model of Ok, Ortoleva and Riella (2015).

[3]We will of course discuss these in detail in Section 4, but let us mention here that the answers are: (1) Yes. (2) Yes. (3) Yes. No. (4) Yes for 2-asset portfolio diversification, but no for 3 (or more)-asset portfolio diversification (even when preferences are quasitransitive). Not only that some of these answers are somewhat unexpected, they are also of economic significance. In particular, the analysis of (2) yields a generalization of Nash's existence theorem, while that of (3) shows that the common experimental practice of deducing preferences of the subjects by asking them "price" the lotteries is suspect (because this practice is not valid even for acyclic preferences). On the other hand, the analysis of (4) shows that the validity of the insight "risk aversion implies preference for diversification," depends on the transitivity hypothesis. It is false even for continuous, complete and quasitransitive preferences that satisfy the Independence Axiom.

The paper is organized as follows. In Section 2 we introduce the premilinary notation and terminology that we adopt throughout the paper. Section 3.1 provides a roadmap that explains how our representation theorems relate to the expected utility theorem. In Section 3.2, we introduce, and discuss in detail, the notion of *coalitional minmax expected utility representation*. Section 3.3 points to the fundamental nature of this notion by showing that any reflexive relation (on a lottery space) which satisfies the Independence Axiom can be represented in this manner. In the remaining subsections of Section 3, we introduce the other rationality properties of expected utility theorem, and refine our representation step by step. (We thus trace the roadmap of Section 3.1 and eventually end up with the expected utility theorem.) Section 4 presents numerous applications in which our representation theorems are used in a substantive manner. In addition to answering (generalizations of) the questions we posed above, we chararacterize in this section stochastically monotonic and risk averse preferences over monetary lotteries which need not be either complete or transitive. We conclude the main body of the paper with Section 5 in which we state a number of open problems that arise from this work in precise terms. Appendix A contains the proofs of our representation theorems, and for completeness of exposition, Appendix B discusses an alternative expected multi-utility representation notion (for discontinuous) preferences.

2 Preliminaries

Order-Theoretic Nomenclature. Let A be a nonempty set. By a **binary relation** on A, we mean a nonempty subset $\mathbf{R}$ of $A \times A$, but, as usual, we often write $a\ \mathbf{R}\ b$ to mean $(a, b) \in \mathbf{R}$. Moreover, for any $a \in A$ and nonempty $B \subseteq A$, we write $a\ \mathbf{R}\ B$ to mean $a\ \mathbf{R}\ b$ for each $b \in B$. (The expression $B\ \mathbf{R}\ b$ is similarly understood.) We denote the asymmetric part of this relation by $\mathbf{R}^{>}$. (That is, $\mathbf{R}^{>}$ is either empty or it is a binary relation on A such that $a\ \mathbf{R}^{>}\ b$ iff $a\ \mathbf{R}\ b$ but not $b\ \mathbf{R}\ a$). The symmetric part of $\mathbf{R}$ is then defined as $\mathbf{R}^{=} := \mathbf{R} \backslash \mathbf{R}^{>}$.

We recall that $\mathbf{R}$ is said to be **reflexive** if $a\ \mathbf{R}\ a$ for each $a \in A$, **irreflexive** if $a\ \mathbf{R}\ a$ is false for every $a \in A$, **complete** (or *total*) if either $a\ \mathbf{R}\ b$ or $b\ \mathbf{R}\ a$ holds for each $a, b \in A$, **antisymmetric** if $a\ \mathbf{R}\ b$ and $b\ \mathbf{R}\ a$ do not both hold for any distinct $a, b \in A$, and **transitive** if $a\ \mathbf{R}\ b$ and $b\ \mathbf{R}\ c$ imply $a\ \mathbf{R}\ c$ for each $a, b, c \in A$. We say that $\mathbf{R}$ is **quasitransitive** if $\mathbf{R}^{>}$ is either empty or transitive, and **acyclic** if for no finitely many elements $a_1, ..., a_k$ in A, we may have $a_1\ \mathbf{R}^{>} \cdots \mathbf{R}^{>}\ a_k\ \mathbf{R}^{>}\ a_1$. It is plain that transitivity of a binary relation implies its quasitransitivity, and its quasitransitivity implies its acyclicity, but not conversely.

Let $\mathbf{R}$ be a binary relation on A. If $\mathbf{R}$ is reflexive and transitive, it is said to be a **preorder** on A. If $\mathbf{R}$ is an antisymmetric preorder on A, we say that $\mathbf{R}$ is a **partial order** on A, and if it is a complete partial order on A, we say that it is a **linear order** on A. If A is understood to be endowed with a particular partial order on A, we may refer to it as a **poset** (short for *partially ordered set*). Similarly, when A is endowed with a linear order on A, we refer to it as a **loset**.

When A is endowed with a topological structure, there are a variety of ways of defining a notion of "continuity" for a binary relation $\mathbf{R}$ on A. We will adopt the most commonly used version of these here, and say that $\mathbf{R}$ is **continuous** if it is a closed subset of $A \times A$. When A is a metric space, this is the same thing as saying that $\lim x_m\ \mathbf{R}\ \lim y_m$ for any two convergent sequences (x_m) and (y_m) in A with x_m $\mathbf{R}$ y_m for each m.[4]

Lotteries. Throughout this paper, we let X stand for a separable metric space which is interpreted as a universal set of (non-random) outcomes. By a **lottery** on X, we mean a Borel probability measure on X. The **support** of a lottery p on X is the smallest closed subset S of X such that $\mathbf{p}(S) = 1$. In turn, by a **simple lottery** on X, we mean a lottery on X with finite support, and by a **degenerate lottery** on X, we mean a lottery whose support is a singleton. As is standard, we denote by δ_ω the degenerate lottery on X whose support is $\{\omega\}$.

The set of all lotteries on X is denoted by $\Delta(X)$. As usual, we think of this set as a topological space relative to the topology of weak convergence; as such this space is metrizable. Throughout the paper, $\mathbf{C}(X)$ stands for the set of all continuous real maps on X, and $\mathbf{C}_\mathrm{b}(X)$ is the set of all continuous and bounded real maps on X. (When X is compact, these two sets are the same.) We always think of $\mathbf{C}_\mathrm{b}(X)$ as a metric space relative to the sup-metric. The expectation of any map u in $\mathbf{C}_\mathrm{b}(X)$ with respect to a probability measure p in $\Delta(X)$ is denoted by $\mathsf{E}(u, p)$, that is,

$$\mathsf{E}(u, p) := \int_X u dp.$$

Affine Relations. For any linear space Y, a function $f : \Delta(X) \to Y$ is said to be **affine** if $f(\lambda p + (1 - \lambda)q) = \lambda f(p) + (1 - \lambda)f(q)$ for every $p, q \in \Delta(X)$ and $\lambda \in [0, 1]$. For example, $\mathsf{E}(u, \cdot)$ is an affine real map on $\Delta(X)$ for any $u \in \mathbf{C}_\mathrm{b}(X)$. Finally, we say that a binary relation $\mathbf{R}$ on $\Delta(X)$ **satisfies the Independence Axiom**, if

$$p \mathbf{R} q \quad \text{iff} \quad \lambda p + (1 - \lambda)r \mathbf{R} \lambda q + (1 - \lambda)r$$

for every $p, q, r \in \Delta(X)$ and $\lambda \in (0, 1]$.

Definition. An **affine relation** on $\Delta(X)$ is a reflexive binary relation on $\Delta(X)$ which satisfies the Independence Axiom.

We focus on such binary relations exclusively in this paper.

3 Main Results

3.1 A Roadmap

Decision theory under risk takes a reflexive binary relation $\mathbf{R}$ on $\Delta(X)$ as its primitive, and interprets it as the preference relation of an individual over lotteries on X. In turn, the classical expected utility theory makes four fundamental assumptions on $\mathbf{R}$, namely, it assumes that $\mathbf{R}$ is complete, transitive, affine, and continuous. That is, expected utility theory focuses at large on a continuous and affine preorder on $\Delta(X)$. The founding result of this theory is the

[4]Some authors use the term "closed-continuity" for what we call continuity of a binary relation. In turn, one may say that $\mathbf{R}$ is *open-continuous* if $\mathbf{R}^>$ is an open subset of $A \times A$. These notions are in general distinct, but they reduce to the same thing when $\mathbf{R}$ is complete.

von Neumann-Morgenstern (*Expected Utility*) *Theorem* which provides an extremely useful characterization for such preorders.

The von Neumann-Morgenstern Theorem.[5] *Let X be a separable metric space and $\succsim$ a binary relation on $\Delta(X)$. Then, $\succsim$ is a continuous, complete and affine preorder on $\Delta(X)$ if, and only if, there is a (utility) function $u \in \mathbf{C}_{\mathrm{b}}(X)$ such that*

$$p \succsim q \quad \textit{iff} \quad \mathbf{E}(u,p) \geq \mathbf{E}(u,q)$$

for every p and q in $\Delta(X)$.

The assumption that separates expected utility theory from other theories of (cardinal) utility is surely the Independence Axiom. As such, it is historically the most widely debated axiom of the theory, and we have little to contribute to this discussion here. Instead, we take this axiom as what "makes" expected utility theory, and attempt to understand the structure it entails for preferences over lotteries in a systematic manner. As the reflexivity property is none other than a triviality, the primitive of this paper will always be an affine relation on $\Delta(X)$, where X is a separable metric space.

Our task is to understand exactly how the Independence Axiom meshes with the three other assumptions that are imposed on $\succsim$ in the above theorem, namely, [C] completeness, [T] transitivity and [CC] continuity. As we discussed in detail in the Introduction, there are good economic reasons for doing so, and some of our applications in Section 4 will justify this further. In Section 3.3, we begin our investigation by providing a result which is in a sense the opposite extreme of the von Neumann-Morgenstern Theorem. This theorem drops all three of the hypotheses [C], [T] and [CC] in that theorem, and characterizes those reflexive preference relations that are known only to satisfy the Independence Axiom. We will see that the main strokes of the von Neumann-Morgenstern Theorem are not lost even at this level of generality. It is just that the notion of "expected utility" is then replaced by a notion of "coalitional expected utility." It then remains further to prove seven theorems, one for each combination of the hypotheses [C], [T] and [CC]. Three of these have already been established in the literature (at least in the case where X is compact). Obviously, if we posit all three of these hypotheses, we get the von Neumann-Morgenstern Theorem. The case where we assume only [C] and [T] was settled (in the form of a "lexicographic expected multi-utility theorem,") by Hausner and Wendel (1952), and the case where we assume only [T] and [CC] was dealt with (in the form of a "vector expected multi-utility theorem,") by Dubra, Maccheroni and Ok (2004). The remaining four cases are settled in Section 3.3.

Of particular interest among the results presented in this paper is one in which only [C] and [CC] are imposed on $\succsim$. This result, given in Section 3.3.6, tells us exactly how the von Neumann-Morgenstern Theorem alters when we drop transitivity from its set of hypotheses (when X is compact). In addition, this result also serves as a stepping stone for determining the impact of weakening the transitivity axiom in the von Neumann-Morgenstern Theorem, as opposed to omitting it entirely. We present two results of this nature here. First, we show how the von Neumann-Morgenstern Theorem modifies if we replace [T] with [QT] quasitransitivity

[5] This theorem has appeared first in an appendix of the 2^{nd} edition of the opus von Neumann and Morgenstern (1947), albeit in the special case where X is finite. To our knowledge, the general version of the theorem as we state here was put on record first in Grandmont (1972).

(Section 3.3.7), and then carry out the same exercise for the case where [T] is replaced with [A] acyclicity (Section 3.3.8). Insofar as optimization- and game-theoretic applications are concerned, we find (in Section 4) that the latter result is surprisingly useful.

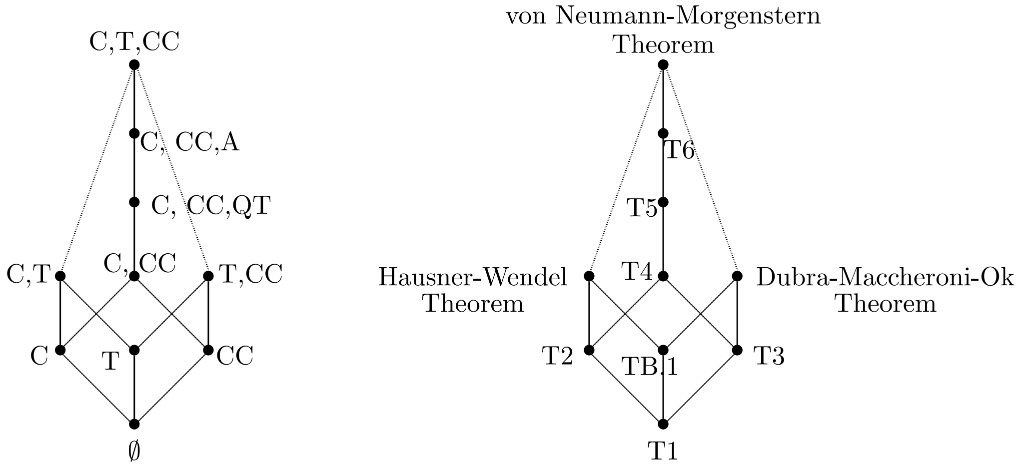

Figure 1

Figure 1 (in which Ti refers to Theorem i below) is meant to illustrate which of our theorems corresponds to which combination of the hypotheses imposed on $\succsim$.[6] As such, this figure is meant to provide a roadmap for Section 3.3.

3.2 Coalitional Expected Utility Representations

All of the representation theorems we present in this paper (with the exception of the treatment given in Appendix B) are couched in terms of a new notion of expected utility representation. This section is devoted to a preliminary discussion of this notion.

3.2.1 Coalitional Minmax Expected Utility Representation

For purposes of interpretation, let us refer to a person whose preference relation on $\Delta(X)$ is a complete, continuous and affine preorder as a "von Neumann-Morgenstern agent." In this jargon, for instance, the von Neumann-Morgenstern Theorem says that the preferences of a von Neumann-Morgenstern agent on $\Delta(X)$ admits an expected utility representation (with a continuous and bounded cardinal utility function). Now consider an individual whose preferences on $\Delta(X)$ are modeled by means of an arbitrarily given affine relation $\mathbf{R}$ on $\Delta(X)$. This individual may not be a von Neumann-Morgenstern agent, because her preference relation may fail to be complete, transitive, and/or continuous. However, it is possible that the preferences of this individual still arise from some type of aggregation of the preferences of several "von Neumann-Morgenstern agents."

[6] Not every theorem we report below works with an arbitrary separable metric space X. But we can still look at these results from a unified perspective, for each of them remains valid when X is compact. Formally speaking, then, one should presume that Figure 1 is based on an affine (preference) relation on $\Delta(X)$, where X is a compact metric space.

To illustrate, suppose there is a collection $\mathcal{U}$ of continuous and bounded real maps on X such that

$$p \mathbf{R} q \quad \text{iff} \quad \mathbf{E}(u,p) \geq \mathbf{E}(u,q) \text{ for each } u \in \mathcal{U} \tag{1}$$

for every p and q in $\Delta(X)$. In this case, it makes good sense to think of our individual as consisting of several "rational selves," each of whom is a von Neumann-Morgenstern agent. The representation maintains that this person prefers a lottery p over another lottery q iff *every one* of her "rational selves" says that p is better than q. This is exactly the notion of expected multi-utility representation advanced by Dubra, Maccheroni and Ok (2004). It corresponds to those (affine) preferences on $\Delta(X)$ which are transitive and continuous, but not necessarily complete. (If there is a disagreement about the desirability of p and q among some of these selves, that is, $\mathbf{E}(u,p) > \mathbf{E}(u,q)$ and $\mathbf{E}(v,q) > \mathbf{E}(v,p)$ for some $u, v \in \mathcal{U}$, then the agent remains indecisive about the ranking of p and q.)

This is of course not the only way of representing $\mathbf{R}$ by means of multiple expected utilities. Another interesting alternative obtains if we choose a collection $\mathcal{V}$ of continuous and bounded real maps on X such that

$$p \mathbf{R} q \quad \text{iff} \quad \mathbf{E}(v,p) \geq \mathbf{E}(v,q) \text{ for some } v \in \mathcal{V} \tag{2}$$

for every p and q in $\Delta(X)$. In this case too we can think of our individual as consisting of several "rational selves," each of whom is a von Neumann-Morgenstern agent. For this individual, it is enough to "justify" a decision from the perspective of only one of her selves, that is, she prefers p over q iff *at least one* of her "rational selves" says that p is better than q. This is very much in the spirit of the justifiable preference representation studied by Lehrer and Teper (2011). It corresponds to those (affine) preferences on $\Delta(X)$ which are complete, quasitransitive and continuous, but not necessarily transitive (due to the potential "thickness" of indifference curves).

What of (affine) preferences on $\Delta(X)$ that may fail to be both complete and transitive? It turns out that we can again think of such preferences as arising from the aggregation of several "rational selves," but the representation notion to be used here must be more general than those in (1) and (2). What we need is to view the individual's preferences as an aggregation of the preferences of *groups* of von Neumann-Morgenstern agents where the preferences of each group itself is determined by aggregating the preferences of its members. To put this precisely, let $\mathbb{U}$ be a collection of nonempty convex subsets of $\mathbf{C}_{\mathrm{b}}(X)$ such that

$$p \mathbf{R} q \quad \text{iff} \quad [\text{for every } \mathcal{U} \in \mathbb{U} \text{ there is a } u \in \mathcal{U} \text{ such that } \mathbf{E}(u,p) \geq \mathbf{E}(u,q)] \tag{3}$$

for every p and q in $\Delta(X)$. This representation generalizes both of the representation notions we considered above. (If every element of $\mathbb{U}$ is a singleton, (3) reduces to (1), and if $\mathbb{U}$ is itself a singleton, then (3) reduces to (2).) And, just as in those cases, it is *as if* our individual has multiple "selves," where each "self" is a von Neumann-Morgenstern agent. But, unlike those cases, her decision making is now guided by *coalitions* of these "selves." More precisely, there is a set $\mathcal{S}$ of coalitions of von Neumann-Morgenstern agents such that

$$\mathbf{R} = \bigcap_{S \in \mathcal{S}} \bigcup \{\succsim_{i,S} : i \in I_S\}, \tag{4}$$

where $\succsim_{i,S}$ is the preference relation of the "self i in the coalition S." (Here, $\mathcal{S}$ is the index set (of the coalitions), and for each $S \in \mathcal{S}$, I_S is the index set (of the members of the coalition S).) Thus, our individual ranks p over q iff in any coalition of her von Neumann-Morgenstern "selves," there is at least one "self" that says p is better than q, or put differently, p **R** q iff no coalition of her totally rational "selves" may block the lottery p in favor of q.

Suppose **R** satisfies (3) for every p and q in $\Delta(X)$. Then, clearly,

$$p \mathbf{R}\, q \quad \text{implies} \quad \inf_{\mathcal{U}\in\mathbb{U}} \sup\{\mathsf{E}(u,p) - \mathsf{E}(u,q) : u \in \mathcal{U}\} \geq 0 \tag{5}$$

for every $p, q \in \Delta(X)$. While the converse of this implication is in general not valid, we will prove in Section 3.3.4 that one can choose $\mathbb{U}$ in such a way that it holds when X is compact and **R** is continuous. What is more, we can do this in a way that allows us to replace the operators inf and sup with those of min and max, respectively. Motivated by these observations, we will refer to the representation notion of (3) in what follows as a *coalitional minmax expected utility representation* for **R**.

Definition. Let **R** be a binary relation on $\Delta(X)$. We say that **R** admits a **coalitional minmax expected utility representation** if there is a collection $\mathbb{U}$ of nonempty convex subsets of $\mathbf{C}_{\mathrm{b}}(X)$ such that (3) holds for every p and q in $\Delta(X)$. In turn, we refer to any such collection $\mathbb{U}$ as a **coalitional minmax expected utility for R**.

The main thesis of the present paper is that this representation notion provides a unifying structure for expected utility theory at large. Indeed, in our first result below we will see that *any* reflexive binary relation on $\Delta(X)$ which satisfies the Independence Axiom admits a coalitional minmax expected utility representation. As we introduce more structure on such a relation, say, in the form of [C], [CC], [T], [QT] and/or [A], the representing collection $\mathbb{U}$ becomes more and more concrete. In particular, the von Neumann-Morgenstern Theorem says that if we impose [C], [CC] and [T] jointly, we can choose $\mathbb{U}$ as a singleton collection that consists of a singleton set.

3.2.2 Coalitional Maxmin Expected Utility Representation

There is a natural alternative to coalitional minmax expected utility representation for affine relations on $\Delta(X)$. Specifically, we can also consider a dual representation notion for **R** which is of "maxmin" form; this is obtained by switching the order of quantifiers in (3). (The resulting representation notion is in the same spirit to what Lehrer and Teper (2011) call "Knightian-justifiable" preferences in the context of preferences over Anscombe-Aumann acts.) This dual representation would ask for a collection $\mathbb{V}$ of nonempty convex subsets of $\mathbf{C}_{\mathrm{b}}(X)$ such that

$$p \mathbf{R}\, q \quad \text{iff} \quad [\text{there is a } \mathcal{V} \in \mathbb{V} \text{ such that } \mathsf{E}(v,p) \geq \mathsf{E}(v,q) \text{ for each } v \in \mathcal{V}] \tag{6}$$

for every p and q in $\Delta(X)$. (In what follows, we refer to this notion as *coalitional maxmin expected utility representation.*) Adopting the jargon and notation introduced above, this is the same thing as saying that

$$\mathbf{R} = \bigcup_{T\in\mathcal{T}} \bigcap \{\trianglerighteq_{i,T} : i \in J_T\}$$

where $\mathcal{T}$ and J_T (for each $T \in \mathcal{T}$) are index sets, and each $\trianglerighteq_{i,T}$ is a continuous and complete affine preorder on X. Thus, the individual whose preferences are modeled by $\mathbf{R}$ prefers p to q iff there is at least one coalition of her von Neumann-Morgenstern "selves" that surely recommends choosing p over q in the sense that every single "self" that belongs to that coalition says p is better than q. The preference relation $\mathbf{R}$ is thus *rationalized* in the same sense that is proposed by Cherepanov, Feddersen and Sandroni (2013), albeit, rationalization takes place here through *coalitions* of "selves" as opposed to the "selves" themselves.[7]

In the abstract, coalitional minmax and maxmin expected utility representations are equivalent. In fact, we can always transform a given set $\mathbb{U}$ (of sets of utility functions) that represents a relation in the former sense into another collection $\mathbb{V}$ that represents the same relation in the latter sense, and conversely.[8] (In particular, the word "minmax" can be replaced with "maxmin" in Theorem 1 below.) However, as we put further conditions on $\mathbf{R}$, say [C], [QT], [A], etc., the mathematical structure of the representing set of sets of utilities exhibits different characteristics depending on which type of coalitional expected utility representation notion is adopted. One should thus make a choice in this regard, and as such, we work exclusively with the notion of coalitional *minmax* expected utility representation in this paper.

3.3 Representation Theorems

3.3.1 Affine Relations

Our first result identifies what remains of the von Neumann-Morgenstern Theorem if we delete the hypotheses of completeness, transitivity and continuity from its statement. Its main purpose is to exhibit the fundamental nature of the notion of coalitional minmax expected utility representation.

Theorem 1. *Let X be a separable metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$. Then, $\mathbf{R}$ is an affine relation on $\Delta(X)$ if, and only if, it admits a coalitional minmax expected utility representation.*

We can thus find a coalitional minmax expected utility $\mathbb{U}$ for *any* affine relation $\mathbf{R}$ on $\Delta(X)$. This is remarkable because every (utility) function in any member of $\mathbb{U}$ is *continuous* and bounded, even though $\mathbf{R}$ itself need not be continuous. This fact may help working with affine preference relations in general. Conversely, Theorem 1 specifies a general method of defining an arbitrary affine relation on $\Delta(X)$. Using a set of sets of continuous and bounded real functions as in (3), apparently, exhausts all such binary relations. We will provide some applications of this observation in Section 4.

[7] Put more accurately, if we agree to refer to a continuous affine preorder on $\Delta(X)$ as a "rationale," and understand from the statement $p \ \mathbf{R} \ q$ that the agent chooses p from the feasible set $\{p, q\}$, then our representation notion reduces to what Cherepanov, Feddersen and Sandroni (2013) call *rationalization* of a choice correspondence (on pairwise sets).

[8] Suppose $\mathbf{R}$ is a relation on $\Delta(X)$ which admits a coalitional minmax expected utility representation $\mathbb{U}$. For any $(p, q) \in \mathbf{R}$ and any $\mathcal{U} \in \mathbb{U}$, pick a function $u_{p,q,\mathcal{U}}$ in $\mathcal{U}$ such that $\mathsf{E}(u_{p,q,\mathcal{U}}, p) \geq \mathsf{E}(u_{p,q,\mathcal{U}}, q)$. Then, for any $(p, q) \in \mathbf{R}$, define $\mathcal{U}_{p,q} := \text{co}\{u_{p,q,\mathcal{U}} : \mathcal{U} \in \mathbb{U}\}$, and finally, set $\mathbb{V} := \{\mathcal{U}_{p,q} : (p, q) \in \mathbf{R}\}$. It is routine to check that (5) holds for every $p, q \in \Delta(X)$; thus $\mathbf{R}$ admits a coalitional maxmin expected utility representation. The converse implication is proved analogously.

3.3.2 Complete Affine Relations

We now ask how the representation obtained in Theorem 1 would modify if we knew that the preferences of the agent were complete. Or put differently, the issue is to understand what happens to the von Neumann-Morgenstern Theorem if we delete the hypotheses of transitivity and continuity from its statement.

Intuitively, given the interpretation of "coalitional minmax expected utility representation," we would expect that the completeness of preferences would imply some sort of consistency across coalitions. (For instance, if $\{\{u\},\{v\}\}$ is a coalitional minmax expected utility for the preference relation $\mathbf{R}$ with $\mathsf{E}(u,p) > \mathsf{E}(u,q)$ and $\mathsf{E}(v,q) > \mathsf{E}(v,p)$, then $\mathbf{R}$ would be indecisive about the ranking of p and q.) Fortunately, there is an easy way of formalizing exactly what sort of consistency must there be across the members of its coalitional minmax expected utility $\mathbb{U}$ when $\mathbf{R}$ is complete. In the setting of Theorem 1, it turns out that completeness of $\mathbf{R}$ would be captured fully if we posit that, given any two coalitions $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{U}$, there is a member of $\mathcal{U}$ and a member of $\mathcal{V}$ whose (von Neumann Morgenstern) preferences are fully aligned. More concretely, but equivalently, we require that any two coalitions in $\mathbb{U}$ overlap, and hence say that a collection $\mathbb{U}$ of nonempty subsets of $\mathbf{C}_\mathrm{b}(X)$ is **coherent** if $\mathcal{U} \cap \mathcal{V} \neq \emptyset$ for every $\mathcal{U}, \mathcal{V} \in \mathbb{U}$. Our next result shows that completeness of $\mathbf{R}$ allows us to choose a coherent coalitional minmax expected utility for it.

Theorem 2. *Let X be a separable metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$. Then, $\mathbf{R}$ is a complete affine relation on $\Delta(X)$ if, and only if, it admits a coherent coalitional minmax expected utility representation.*

The interpretation of this result is identical to that of Theorem 1, with the proviso that $\mathbb{U}$ has now a bit more structure (in the form of coherence). However, thanks to its completeness, we now have a nice characterization of the *strict* part of the preference relation $\mathbf{R}$ as well. Indeed, if $\mathbf{R}$ is complete and $\mathbb{U}$ is a coalitional minmax expected utility for $\mathbf{R}$, then

$$p \; \mathbf{R}^{>} \; q \quad \text{iff} \quad [\text{there is a } \mathcal{U} \in \mathbb{U} \text{ such that } \mathsf{E}(u,p) > \mathsf{E}(u,q) \text{ for each } u \in \mathcal{U}] \tag{7}$$

for every p and q in $\Delta(X)$. In terms of our "coalitional representation" interpretation, therefore, an individual who has a complete (but possibly non-transitive and discontinuous) preference relation on $\Delta(X)$ which satisfies the Independence Axiom *strictly* prefers a lottery p over q iff each of her von Neumann-Morgenstern "selves" in at least one of her (mental) coalitions likes p *strictly* better than q. When $p \; \mathbf{R}^{>} \; q$, then, choice of q over p would surely be blocked by that coalition (in the "mind" of the agent).

3.3.3 (Complete) Affine Preorders

Let $\mathbf{R}$ be an affine relation on $\Delta(X)$. Then, by Theorem 1, there is a coalitional minmax expected utility $\mathbb{U}$ for $\mathbf{R}$. If we assume further that $\mathbf{R}$ is transitive, $\mathbb{U}$ must have a particular structure. Unfortunately, unlike the case we considered in the previous section, we were unable to find an applicable way to characterize this structure. We presently leave this matter as an open problem. (See Section 5.)

It is worth noting here that there is another approach that one can adopt for studying affine preorders. Indeed, complete and affine partial orders (on an arbitrary mixture space) were studied in the earlier literature. In particular, Hausner and Wendel (1952) have proved a rather deep characterization theorem for such partial orders in the form of a lexicographic multi-utility representation. It is trivial to extend this result to the case of affine preorders on $\Delta(X)$ where X is any metric space. With a little bit more effort, one can even deduce a version of it that applies in the absence of the completeness hypothesis, thereby providing a *coalitional* lexicographic expected multi-utility theorem. While they are couched in terms of a different representation notion for affine preorders than the one we focus on here, these findings nevertheless accord fully with the primary objective of the present paper. Not to deter from the unified perspective of the exposition, but still to provide a comprehensive coverage, we thus postpone the related discussion to Appendix B which contains a precise statement of the Hausner-Wendel Theorem, and a formalization of how one may drop the completeness hypothesis in the statement of that theorem.

3.3.4 Continuous Affine Relations

We now turn to the continuous (and reflexive) preference relations on $\Delta(X)$ which satisfy the Independence Axiom. This is the problem of determining how the von Neumann-Morgenstern Theorem would modify if we deleted the hypotheses of completeness and transitivity from its statement.

We again take the "coalitional (multi-utility) representation" as our target. As such, the question is to find out what sort of a structure continuity would entail for the coalitional minmax expected utility $\mathbb{U}$ found in Theorem 1. At least when X is compact, there is quite a nice answer to this question. In the context of that theorem, but with X compact, continuity of $\mathbf{R}$, which is often viewed in decision theory as a technical, but duly reasonable, hypothesis, allows us to guarantee that every element of $\mathbb{U}$ is a *compact* collection of continuous and bounded (utility) functions on X. In fact, we can do a bit better than this. In this case we can even choose $\mathbb{U}$ itself as a *compact* collection (relative to the Hausdorff metric).[9]

Theorem 3. *Let X be a compact metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$. Then, $\mathbf{R}$ is a continuous affine relation on $\Delta(X)$ if, and only if, there is a compact collection $\mathbb{U}$ of nonempty compact and convex subsets of $\mathbf{C}(X)$ such that (3) holds for every p and q in $\Delta(X)$.*

The interpretation of this result is identical to that of Theorem 1, but now $\mathbb{U}$ has some topological structure. In particular, this allows us to express the representation (3) alternatively as

$$p \mathbf{R} q \quad \text{iff} \quad \min_{\mathcal{U}\in\mathbb{U}} \max_{u\in\mathcal{U}} (\mathbf{E}(u,p) - \mathbf{E}(u,q)) \geq 0$$

for every p and q in $\Delta(X)$. This justifies the "minmax" in the term "coalitional minmax expected utility representation."

[9] Let Z be a metric space, and denote the set of all nonempty compact subsets of Z by $\mathbf{k}(Z)$. We denote the Hausdorff metric, and convergence with respect to this metric, on $\mathbf{k}(Z)$ by d^{H} and $\to^{\mathrm{H}}$, respectively. We recall that d^{H} is a uniform metric in the sense that $d^{\mathrm{H}}(S,T) = \sup\{|d(z,S) - d(z,T)| : z \in Z\}$ for any nonempty compact subsets S and T of Z, where d is the metric of the underlying space Z. (In the present discussion, Z is $\mathbf{C}(X)$, and d is the sup-metric.)

3.3.5 Continuous Affine Preorders

Of the remaining two variations of the von Neumann-Morgenstern Theorem, one concerns the identification of the structure of continuous, reflexive and transitive preference relations that satisfy the Independence Axiom. Starting with the pathbreaking work of Aumann (1962), this case has actually received quite a bit of attention in the literature. In particular, the following theorem of Dubra, Maccheroni and Ok (2004) identifies exactly what happens if we drop the completeness hypothesis in the statement of the von Neumann-Morgenstern Theorem (when X is compact).

The Dubra-Maccheroni-Ok Theorem. *Let X be a compact metric space and $\succsim$ a binary relation on $\Delta(X)$. Then, $\succsim$ is a continuous and affine preorder on $\Delta(X)$ if, and only if, there exists a compact subset $\mathcal{U}$ of $\mathbf{C}(X)$ such that*

$$p \succsim q \quad \textit{iff} \quad \mathbf{E}(u,p) \geq \mathbf{E}(u,q) \textit{ for each } u \in \mathcal{U}$$

for every p and q in $\Delta(X)$.[10]

This theorem is sometimes referred to as the "expected multi-utility theorem" in the literature. As it is discussed in Dubra, Maccheroni and Ok (2004) in detail, we will not elaborate on it here. Suffice it to say that the coalitional minmax expected utility representation (3) reduces to the representation of the Dubra-Maccheroni-Ok Theorem if each element of $\mathbb{U}$ is a singleton.[11]

3.3.6 Complete and Continuous Affine Relations

We now ask what happens if we drop only the hypothesis of transitivity in the statement of the von Neumann-Morgenstern Theorem. Given Thereoms 2 and 3, there is a straightforward conjecture in this regard. Let $\mathbf{R}$ be an affine relation on $\Delta(X)$, where X is compact. By Theorem 1, we know that there is a coalitional minmax expected utility $\mathbb{U}$ for $\mathbf{R}$. Theorem 2 says that we can choose $\mathbb{U}$ as coherent, provided that $\mathbf{R}$ is complete, and Theorem 3 says that we can choose it as a compact collection of compact sets (of cardinal utility functions), provided that $\mathbf{R}$ is continuous. Thus, it is only natural to conjecture that we can choose $\mathbb{U}$ as a coherent and compact collection of compact and convex sets, provided that $\mathbf{R}$ is both complete and continuous. While this is by no means an immediate consequence of Theorems 2 and 3, it is nevertheless correct. This is the content of our next result.

Theorem 4. *Let X be a compact metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$. Then, $\mathbf{R}$ is a continuous and complete affine relation on $\Delta(X)$ if, and only if, there is a coherent and compact collection $\mathbb{U}$ of nonempty compact and convex subsets of $\mathbf{C}(X)$ such that* (3) *holds for every p and q in $\Delta(X)$.*[12]

[10] It is known that compactness cannot be relaxed to separability in this theorem (unless we choose the cardinal utilities as Borel measurable instead of continuous). See Evren (2008) for details.

[11] The "only if" part of the above theorem is a bit sharper than the original statement of the expected multi-utility theorem in that we are now able to guarantee that the set $\mathcal{U}$ of utilities is in fact compact (relative to the sup norm).

[12] In this theorem, we can take $\mathbb{U}$ as countable instead of compact, while in Theorem 3 we can take $\mathbb{U}$ as countable *and* compact. In fact, it is these stronger assertions that we prove in Appendix A.

This result provides a complete characterization of preferences over lotteries (with a compact prize space) where all hypotheses of expected utility theory, with the exception of transitivity, hold. As such, it provides an exhaustive method of constructing non-transitive, but complete and continuous, preferences that satisfy the Independence Axiom.

3.3.7 Justifiable Preferences

It is natural at this point to inquire how we may strengthen the representation obtained in Theorem 4 further if we introduce some "partial" transitivity properties into the model. In this and the next section, we look at this issue by means of the properties of *quasitransitivity* and *acyclicity*. While the former property is suitable if we wish to model a situation in which preference cycles arise only due to "nontransitive indifference," the latter property is essential for optimization-theoretic exercises.[13]

Let us begin by noting that, in the context of Theorem 4, quasitransitivity property is equivalent to the apparently more basic property of *strict convexity*. Formally, we say that a binary relation $\mathbf{R}$ on $\Delta(X)$ is **strictly convex** if

$$\{p, q\} \ \mathbf{R}^{>} \ r \quad \text{implies} \quad \tfrac{1}{2}p + \tfrac{1}{2}q \ \mathbf{R}^{>} \ r$$

for any $p, q, r \in \Delta(X)$. Clearly, this property has more of the flavor of the Independence Axiom, and its normative justification is self-evident. Moreover, as the following result demonstrates, it refines the representation we obtained in Theorem 4 in an appealing manner.

Theorem 5. *Let X be a compact metric space and $\succsim$ a binary relation on $\Delta(X)$. Then, $\mathbf{R}$ is a continuous, complete and strictly convex affine relation on $\Delta(X)$ if, and only if, there exists a compact and convex subset $\mathcal{V}$ of $\mathbf{C}(X)$ such that*

$$p \ \mathbf{R} \ q \quad \text{iff} \quad \mathbf{E}(v, p) \geq \mathbf{E}(v, q) \text{ for some } v \in \mathcal{V}$$

for every p and q in $\Delta(X)$.

This representation notion complements that found in the Dubra-Maccheroni-Ok Theorem. The latter is the special case of our minmax "coalitional multi-utility representation" where each (of the possibly many) coalitions has a single member, while the representation above corresponds to the case where there is a single coalition (of possibly many members).

The characterization identified in Theorem 5 also relates closely to the recent work of Lehrer and Teper (2011). In that paper, for preferences over Anscombe-Aumann acts (horse race lotteries), an axiomatic characterization is obtained in terms of multiple priors, but a single cardinal utility function. The representation maintains that an act is preferred to another iff for at least one prior the (subjective) expected utility of the former is at least as large as that of the latter. Such preferences, which are aptly called *justifiable preferences* by Lehrer and Teper (2011), are not transitive (but they are quasitransitive), and possess an analogous structure to the one we obtained in Theorem 5. In fact, it makes good sense to think of the preferences characterized in Theorem 5 as the counterpart of Lehrer and Teper's justifiable preferences in the context of risk.[14]

[13] In the case of more concrete scenarios, one may consider other types of "partial" transitivity properties. We look at two such properties in the context of monetary lotteries in Section 4.5.3.

[14] Lehrer and Teper (2011) does not provide such a counterpart, because they assume in their model that

3.3.8 Acyclic Preferences over Lotteries

Even a finite set may not possess a maximal element with respect to a nontransitive binary relation; this is but one of the main difficulties with working with such relations. However, one does not need the full power of transitivity to escape this difficulty. In particular, it is well-known that every nonempty finite subset of a given nonempty set A has a maximal element with respect to a binary relation on A iff that relation is acyclic. As such, it makes sense to think of acyclicity as the strongest relaxation of transitivity which is suitable for optimization theoretic applications.

With this motivation in the background, we now ask how the representation obtained in Theorem 4 would alter if we added acyclicity to its hypotheses. There is actually a rather pleasant answer to this query. Recall that Theorem 4 shows that adding completeness to the setting of Theorem 3 allows us to choose a coherent coalitional minmax expected utility $\mathbb{U}$ for the affine relation $\mathbf{R}$ under consideration. By definition, this means that any two elements of $\mathbb{U}$ have a nonempty intersection. It turns out that if, in addition, we posit acyclicity, then we can choose $\mathbb{U}$ in such a way that *all* of its members intersect.

We say that a collection $\mathbb{U}$ of nonempty subsets of $\mathbf{C}_{\mathrm{b}}(X)$ is **strongly coherent** if $\bigcap \mathbb{U} \neq \emptyset$. Our next result shows that, in the context of Theorem 4, acyclicity of $\mathbf{R}$ is equivalent to the strong coherence of $\mathbb{U}$.

Theorem 6. *Let X be a compact metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$. Then, $\mathbf{R}$ is a continuous, complete and acyclic affine relation on $\Delta(X)$ if, and only if, there is a strongly coherent and compact collection $\mathbb{U}$ of nonempty compact and convex subsets of $\mathbf{C}(X)$ such that* (3) *holds for every p and q in $\Delta(X)$.*

This characterization is likely to be useful in optimization-theoretic applications of expected utility theory. We will provide several such applications in Section 4.

Remark. If we relax the compactness requirement to separability, and omit the continuity hypothesis in this result, the same characterization remains valid, except that instead of strong coherence, we would then have that any finitely many elements of $\mathbb{U}$ overlap. □

4 Applications

4.1 On Constructions of Affine Preferences

The representation theorems we presented in Section 3.3 provide exhaustive methods of constructing certain types of affine relations. These methods may be helpful in settling problems about such preferences which may be difficult to tackle in the abstract. To illustrate, let us recall the well-known facts that a quasitransitive binary relation need not be transitive, and an acyclic binary relation need not be quasitransitive. But is this still the case for complete and continuous affine relations on a lottery space? In other words, is it the case that transitivity and quasitransitivity and/or quasitransitivity and acyclicity are equivalent properties

preferences over constant acts are transitive. In fact, if there is only one state in that model (so the preferences are over $\Delta(X)$), their characterization theorem reduces to the von Neumann-Morgenstern Theorem.

for complete and continuous affine binary relations? It turns out that the answer is no on both accounts, but it does not seem to be easy to construct the sought counterexamples from the primitives. Yet, the representations we found for such relations in Sections 3.3.7 and 3.3.8 make the problem a walk in the park.

Example 1. Let u and v stand for the maps $t \mapsto t$ and $t \mapsto t^2$ on $[0,1]$, and let $\mathbf{R}$ be the binary relation on $\Delta[0,1]$ which is represented by $\mathcal{V} := \text{co}\{u, v\}$ as in Theorem 5. Then, $\mathbf{R}$ is a continuous, quasitransitive and affine relation on $\Delta[0,1]$, and we have

$$p \ \mathbf{R} \ q \quad \text{iff} \quad \text{either } \mathsf{E}(u,p) \geq \mathsf{E}(u,q) \text{ or } \mathsf{E}(v,p) \geq \mathsf{E}(v,q)$$

for any $p, q \in \Delta[0,1]$. This relation is not transitive. For instance, if p and r are degenerate lotteries which pay 0.5 and 0.6, respectively, and q is the lottery that pays 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$, we have $p \ \mathbf{R} \ q \ \mathbf{R} \ r$ and yet $r \ \mathbf{R}^{>} \ p$. □

We can similarly use the representation obtained in Theorem 6 to construct a simple example which shows that a continuous, acyclic and complete affine relation need not be quasitransitive. Instead, however, we will obtain this fact as an immediate consequence of a more general observation in the next section.

4.2 The Preference Reversal Phenomenon

Among the many experimental observations that refute the basic premises of expected utility theory, a particularly striking one is the so-called *preference reversal* (*PR*) *phenomenon*. This phenomenon was first demonstrated by Slovic and Lichtenstein (1968), and then explored by Grether and Plott (1979) in meticulous detail. Let us formalize the PR-phenomenon in terms of lotteries on $[0,1]$. Let $\mathbf{R}$ be a binary relation on $\Delta[0,1]$ which models the preferences of an individual over such lotteries. For any $p \in \Delta[0,1]$, we define

$$S_{\mathbf{R}}(p) := \inf\{a \in [0,1] : \delta_a \ \mathbf{R}^{>} \ p\},$$

the **minimum selling price of** p for the individual. Now, for any real numbers m, M, α and β in $(0,1)$ with $M > m$ and $\alpha > \beta$, consider the lotteries

$$p_{m,\alpha} := \alpha\delta_m + (1-\alpha)\delta_0 \quad \text{and} \quad p_{M,\beta} := \beta\delta_M + (1-\beta)\delta_0.$$

(So, $p_{m,\alpha}$ is a lottery that yields a "small" return with high probability, and $p_{M,\beta}$ one that yields a "large" return with small probability.) We say that **R exhibits the preference reversal (PR) phenomenon** if there exist such numbers m, M, α and β such that $p_{m,\alpha} \ \mathbf{R}^{>} \ p_{M,\beta}$ and yet $S_{\mathbf{R}}(p_{m,\alpha}) < S_{\mathbf{R}}(p_{M,\beta})$. The experimental works we mentioned above have found that the preferences of many subjects indeed exhibit this phenomenon (for some choice of m, M, α and β).

Intuitively speaking, the PR-phenomenon suggests an intrinsic nontransitivity in the evaluation of lotteries by an individual.[15] Indeed, if $\mathbf{R}$ is quasitransitive, then for any two lotteries

[15] While later studies by Holt (1986), Karni and Safra (1987), Segal (1988), and Safra et al. (1990) have shown that the problem may in fact be the violation of a certain aspect of the independence axiom, the experimental works of Tversky et al. (1990) and Loomes et al. (1991) have shown that the PR-phenomenon indeed points to something deeper than this.

p and q on $[0,1]$ with $p\ \mathbf{R}^{>}\ q$, we have $\{a : \delta_a\ \mathbf{R}^{>}\ p\} \subseteq \{a : \delta_a\ \mathbf{R}^{>}\ q\}$ and hence $S_{\mathbf{R}}(p) \geq S_{\mathbf{R}}(q)$. Thus, any "explanation" of choice data of the form "$p\ \mathbf{R}^{>}\ q$ and $S_{\mathbf{R}}(p) < S_{\mathbf{R}}(q)$" necessitates a nontransitive model of preferences over risk. In fact, one of the major motivations behind the SSB utility theory was precisely this fact; Fishburn (1984) has shown that the PR-phenomenon is consistent with this utility theory. But recall that the SSB utility theory relaxes both the transitivity and independence hypotheses of expected utility theory. A natural question is if the PR-phenomenon is consistent with nontransitive risk preferences that are otherwise rational. And even if this is the case, how unrealistic would such a model be?

More concretely, we ask: Can a continuous, reflexive, complete, and acyclic preference relation which satisfies the Independence Axiom (and which is consistent with first order stochastic dominance) exhibit the PR-phenomenon? It is somewhat intuitive to expect that the answer would be no, because such a preference relation is endowed with many traits of rationality. Its only lack of rationality arises from its potential nontransitivity, but that is tamed as well, for this relation can never entail strict cycles. At any rate, it seems quite difficult to answer the question from the primitives. The representation obtained in Theorem 6, on the other hand, provides ample structure for such preferences, and this allows us to settle the matter fairly easily. And, to our surprise, the answer turns out to be affirmative.

Observation. *For any real numbers m, M, α and β in $(0,1)$ with $M > m$ and $\alpha > \beta$, there is a continuous, complete and acyclic affine relation $\mathbf{R}$ on $\Delta[0,1]$ such that $p_{m,\alpha}\ \mathbf{R}^{>}\ p_{M,\beta}$ and yet $S_{\mathbf{R}}(p_{m,\alpha}) < S_{\mathbf{R}}(p_{M,\beta})$.* To prove this, take any such numbers m, M, α and β, and fix some t in $(0,m)$. We consider the continuous real maps u, v and w on $[0,1]$ whose graphs are depicted in Figure 2. Let $\mathbf{R}$ be the binary relation for which $\mathbb{U} := \{\text{co}\{u,v\},\text{co}\{v,w\}\}$ is a coalitional minmax expected utility. We know from Theorem 6 that this is a continuous, complete and acyclic affine relation on $\Delta[0,1]$.[16] Notice that $\mathsf{E}(u,p_{m,\alpha}) = \alpha = \mathsf{E}(v,p_{m,\alpha})$ and $\mathsf{E}(w,p_{m,\alpha}) = \frac{\alpha\beta}{2}$, while the expectation of any of these functions with respect to $p_{M,\beta}$ is β. By the representation, therefore, we see that $p_{m,\alpha}\ \mathbf{R}^{>}\ p_{M,\beta}$. On the other hand, by the representation, $S_{\mathbf{R}}(p_{M,\beta})$ is the infimum of the set $\{a :$ either $(u(a) > \beta$ and $v(a) > \beta)$ or $(v(a) > \beta$ and $w(a) > \beta)\}$. It follows that

$$S_{\mathbf{R}}(p_{M,\beta}) = \min\{\max\{u^{-1}(\beta), v^{-1}(\beta)\}, \max\{v^{-1}(\beta), w^{-1}(\beta)\}\} = \min\{u^{-1}(\beta), w^{-1}(\beta)\},$$

and hence, $S_{\mathbf{R}}(p_{M,\beta}) > t$. But a similar computation shows that

$$S_{\mathbf{R}}(p_{m,\alpha}) = \min\{u^{-1}(\alpha), w^{-1}(\tfrac{\alpha\beta}{2})\} = w^{-1}(\tfrac{\alpha\beta}{2}) < t,$$

and our assertion is proved. □

Corollary. *A continuous, acyclic and complete affine relation on $\Delta[0,1]$ need not be quasitransitive.*

[16] It is readily checked that we have $p\ \mathbf{R}\ q$ whenever p first-order stochastically dominates q. (We can also replace the maps u, v and w by strictly increasing functions in this example so as to guarantee that $p\ \mathbf{R}^{>}\ q$ whenever p strictly first-order stochastically dominates q.) In fact, $\mathbf{R}$ is even transitive with respect to the first-order stochastic dominance in the sense that if p first-order stochastically dominates q and $q\ \mathbf{R}\ r$, we have $p\ \mathbf{R}\ r$. (We call such $\mathbf{R}$ *FSD-transitive*, and provide a characterization for them in Section 4.5.3.)

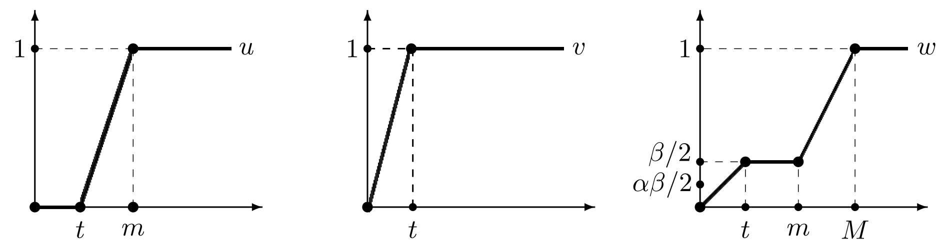

Figure 2

While our main objective here is to demonstrate the potential "use" of Theorem 6, we should note that this observation is of economic substance. About the PR-phenomenon, Grether and Plott (1979), p.623, says that "... this behavior is not simply a violation of some type of expected utility hypothesis. The preference measured one way is the *reverse* of preference measured another and seemingly theoretically compatible way. If indeed preferences exist and if the principle of optimization is applicable, then an individual would place a higher reservation price on the object he prefers. The behavior as observed appears to be simply inconsistent with this basic theoretical proposition." The observation above shows that this "basic theoretical proposition" is misleading at best. Indeed, the preference relation we considered above is continuous, complete and acyclic, and hence the principle of optimization is very much applicable to it. (See Section 4.3.) Moreover, it is rational in every other sense; it is consistent with stochastic dominance and the Independence Axiom. Apparently, contrary to the view of Grether and Plott (1979), the PR-phenomenon is simply a violation of an expected utility hypothesis, namely, transitivity. But, curiously, this violation is not excessive, in that the PR-behavior is consistent with the hypothesis of "acyclic choice."

4.3 Optimization with Acyclic Preferences

It is well-known (and easily proved) that a maximum exists in any given nonempty finite subset of a nonempty set A with respect to a binary relation on A if (and only if) that relation is complete and acyclic. There are numerous results in optimization theory (with far reaching applications in the context of, say, general equilibrium theory) which extends this observation to the case of compact sets and continuous, complete and acyclic relations. The following result is very much in this spirit, but it works with continuous, complete and acyclic affine relations. Providing a direct proof for this result seems (to the authors) quite difficult, but the representation of such relations we found in Section 3.3.8 makes the required argument rather straightforward.

Proposition 1. *Let X be a compact metric space and $\mathbf{R}$ a continuous, complete and acyclic affine relation on $\Delta(X)$. Then, any nonempty closed subset P of $\Delta(X)$ possesses an extreme point which is a maximum element of P with respect to $\mathbf{R}$.*[17]

[17] A slight modification of the proof yields the following more general statement: Let X be a metric space and $\mathbf{R}$ a continuous, complete and acyclic affine relation on $\Delta(X)$. Then, a nonempty closed subset P of $\Delta(X)$ possesses an extreme point which is a maximum element of P with respect to $\mathbf{R}$, provided that $\bigcup\{\text{supp}(p) : p \in P\}$ is relatively compact.

Proof. By Theorem 6, there is a strongly coherent collection $\mathbb{U}$ of nonempty compact subsets of $\mathbf{C}(X)$ such that (3) holds for every p and q in $\Delta(X)$. Take any nonempty closed subset P of $\Delta(X)$, and pick an arbitrary element u of $\bigcap \mathbb{U}$. It is well-known that compactness of X implies that we can view $\Delta(X)$ as a nonempty compact subset of a locally convex topological linear space (namely, ca(X)); see the opening of Appendix A for details. Moreover, $\mathbf{E}(u, \cdot)$ is the restriction of a continuous linear functional on this space. By the Extreme Point Theorem, therefore, there is an extreme point p^* of P such that $\mathbf{E}(u, p^*) \geq \mathbf{E}(u, p)$ for each $p \in P$. But p^* is a maximum element of P with respect to $\mathbf{R}$, for otherwise, that is, if $p \, \mathbf{R}^> \, p^*$ for some $p \in P$, there would exist a $\mathcal{V} \in \mathbb{U}$ such that $\mathbf{E}(v, p) > \mathbf{E}(v, p^*)$ for each $v \in \mathcal{V}$, which is impossible since $u \in \mathcal{V}$. ■

For any binary relation $\mathbf{R}$ on $\Delta(X)$, let us denote the set of all maximal and maximum elements of a given nonempty subset P of $\Delta(X)$ by $\text{MAX}(P, \mathbf{R})$ and $\max(P, \mathbf{R})$, respectively.[18] The Bergstrom-Walker Theorem says that every nonempty compact metric space A has a maximal element with respect to a reflexive and acyclic binary relation on A whose asymmetric part is open in $A \times A$.[19] An immediate application of this result shows that $\max(P, \mathbf{R}) \neq \emptyset$ under the conditions of Proposition 1. (Indeed, affinity of $\mathbf{R}$ plays no role in securing this conclusion.) The novelty of Proposition 1 is instead to show that at least one member of $\max(P, \mathbf{R})$ must come from the extreme points of P (even when P is not convex). Clearly, this would be useful in linear programming type problems in which optimization takes place with respect to an acyclic binary relation on $\Delta(X)$ which satisfies the conditions of Proposition 1. We provide two illustrations.

Example 2. Let X be a metric space and $\mathbf{R}$ a continuous, complete and acyclic affine relation on $\Delta(X)$. Take any nonempty compact subset Y of X. Then, by the Bergstrom-Walker Theorem, there is a maximum element of $\Delta(Y)$ with respect to $\mathbf{R}$. On the other hand, Proposition 1 says that there is an extreme point of $\Delta(Y)$ which belongs to $\max(\Delta(Y), \mathbf{R})$. But it is well-known that any extreme point of $\Delta(Y)$ is a degenerate lottery on Y. Thus, Proposition 1 allows us to conclude that there is a $y \in Y$ such that $\delta_y \, \mathbf{R} \, p$ for every lottery p on Y.[20] □

Example 3. Let $\mathbf{R}$ be a continuous, complete and acyclic affine relation on $\Delta[0, 1]$. Take any positive integer m, and real numbers $\alpha_1, ..., \alpha_m$. Suppose we wish to find a best lottery p on X with respect to $\mathbf{R}$ such that the ith moment of p equals α_i for each $i \in [m]$.[21] To avoid trivialities, let us assume that there is at least one such p, so

$$P := \left\{ p \in \Delta[0,1] : \int_{[0,1]} t^i p(dt) = \alpha_i \text{ for each } i \in [m] \right\}$$

[18] That is, $\text{MAX}(P, \mathbf{R}) := \{p \in P : q \, \mathbf{R}^> \, p \text{ for no } q \in P\}$ and $\max(P, \mathbf{R}) := \{p \in P : p \, \mathbf{R} \, P\}$.

[19] See, Bergstrom (1975) and Walker (1977).

[20] This result is a triviality when $\mathbf{R}$ is known to be transitive, for then there is sure to be a maximum element of $\{\delta_y : y \in Y\}$ with respect to $\mathbf{R}$, and the affinity *and* transitivity of $\mathbf{R}$ allow us to conclude that this element must belong to $\max(\Delta(Y), \mathbf{R})$. The problem is that the second step of this argument fails when all we know is that $\mathbf{R}$ is acyclic. In that case the claim is hardly trivial, but it is nevertheless an easy consequence of Proposition 1 (and hence of Theorem 6).

[21] *Notational Convention.* We write $[m]$ to denote the set $\{1, ..., m\}$ for any positive integer m.

is nonempty. Now notice that $P = \Delta[0,1] \cap H_1 \cap \cdots \cap H_m$ where $H_i := \{\mu \in \text{ca}[0,1] : \int_{[0,1]} t^i \mu(dt) = \alpha_i\}$, which is a closed hyperplane in ca$[0,1]$. As $\Delta[0,1]$ is a convex and compact subset of ca$[0,1]$ (in the weak* topology), a well-known theorem of convex analysis says that any one extreme point of P can be expressed as a convex combination of at most $m+1$ many extreme points of $\Delta[0,1]$.[22] But it is well-known that any extreme point of $\Delta[0,1]$ is a degenerate lottery on $[0,1]$. In view of Proposition 1, we may thus conclude that there is at least one best lottery p on X with respect to $\mathbf{R}$ such that the ith moment of p equals α_i for each $i \in [m]$, and that the support of p has at most $m+1$ elements. □

In passing, we note that there is a way of "computing" $\max(P, \mathbf{R})$ by using the coalitional minmax expected utility representation of $\mathbf{R}$, at least when P is convex. To see this, let X, $\mathbf{R}$ and P be as in Proposition 1, but assume now that P is convex as well. Let $\mathbb{U}$ be a strongly coherent set of nonempty compact subsets of $\mathbf{C}(X)$ which is a coalitional minmax expected utility for $\mathbf{R}$, and for each $\mathcal{U}$ in $\mathbb{U}$, define the binary relation $\succ_{\mathcal{U}}$ on $\Delta(X)$ by $p \succ_{\mathcal{U}} q$ iff $\mathsf{E}(u,p) > \mathsf{E}(u,q)$ for every $u \in \mathcal{U}$. Clearly, $\succ_{\mathcal{U}}$ is an irreflexive and transitive binary relation on $\Delta(X)$ which is open in $\Delta(X) \times \Delta(X)$, and hence, given that P is convex, Proposition 1 of Evren (2014) says that $\text{MAX}(P, \succ_{\mathcal{U}})$ equals $\bigcup_{u \in \mathcal{U}} \arg\max\{\mathsf{E}(u,p) : p \in P\}$, for each $\mathcal{U}$ in $\mathbb{U}$. But, since $\mathbf{R}$ is complete, we have $\max(P, \mathbf{R}) = \text{MAX}(P, \mathbf{R}^>)$, while it is easy to check that a lottery p belongs to $\text{MAX}(P, \mathbf{R}^>)$ iff $p \in \text{MAX}(P, \succ_{\mathcal{U}})$ for every $\mathcal{U} \in \mathbb{U}$. In sum,

$$\max(P, \mathbf{R}) = \bigcap_{\mathcal{U} \in \mathbb{U}} \bigcup_{u \in \mathcal{U}} \arg\max\{\mathsf{E}(u,p) : p \in P\},$$

a formula which allows one to compute $\max(P, \mathbf{R})$ by using the coalitional minmax expected utility $\mathbb{U}$ for $\mathbf{R}$.

4.4 Existence of Nash Equilibrium with Acyclic Preferences

Take any positive integer n, and let X_i be a compact metric space for each $i \in [n]$. We envision a strategic situation in which X_i is the (pure) *action space* of player i, and hence the (pure) *outcome space* is $X := X_1 \times \cdots \times X_n$. Each player is allowed to use mixed strategies, that is, the (mixed) *action space* of player i is $\Delta(X_i)$, and hence the (mixed) *outcome space* is the set of all product Borel probability measures $\bigotimes^n p_i$ on X. But, by contrast to standard game theory, we do not subscribe to the expected utility hypothesis here. Instead, we postulate that the preferences of each player i is given in the form of a continuous and complete affine relation $\mathbf{R}_i$ on $\Delta(X)$. In turn, we say that an n-vector $(p_1^*, ..., p_n^*)$ of lotteries on X is a **(mixed strategy) Nash equilibrium** of $\mathcal{G} := \{X_i, \mathbf{R}_i\}_{i \in [n]}$ if

$$\bigotimes\nolimits^n (p_i^*, p_{-i}^*) \ \mathbf{R}_i \ \bigotimes\nolimits^n (p_i, p_{-i}^*) \quad \text{for every } p_i \in \Delta(X_i) \text{ and } i \in [n].^{23} \tag{8}$$

The set of all (mixed strategy) Nash equilibria of $\mathcal{G}$ is denoted by $NE(\mathcal{G})$.

Clearly, this specification generalizes the classical notion of (mixed strategy) Nash equilibrium. If each $\mathbf{R}_i$ is transitive, then the von Neumann-Morgenstern Theorem applies, and we

[22] As for the definition of ca$[0,1]$, we refer the reader to the preminaries section of Appendix A. We also note that $\Delta[0,1]$ can be replaced with any nonempty, convex and compact subset of ca$[0,1]$ in the argument so far.

[23] We adopt the standard notation of game theory here. For instance, $\bigotimes^n (p_1, p_{-1}^*)$ stands for the product Borel probability measure $p_1 \otimes p_2^* \otimes \cdots \otimes p_n^*$, and so on.

recover the standard situation. And, in that case, a fundamental theorem of game theory, due to the late John Nash, says that $NE(\mathcal{G})$ is not empty. The point of the present application is that we do not need the full power of transitivity to guarantee the existence of equilibrium; it turns out that the acyclicity of each $\mathbf{R}_i$ is enough for this purpose.

Proposition 2. *Take any positive integer n, and for each $i \in [n]$, let X_i be a compact metric space and $\mathbf{R}_i$ a continuous, complete and acyclic affine relation on $\Delta(X)$. Then, there exists a (mixed strategy) Nash equilibrium of $\mathcal{G} := \{X_i, \mathbf{R}_i\}_{i\in[n]}$.*

Proof. By Theorem 6, for each $i \in [n]$ there is a strongly coherent collection $\mathbb{U}(i)$ of nonempty compact subsets of $\mathbf{C}(X)$ such that

$$\bigotimes^n p_j \ \mathbf{R}_i \ \bigotimes^n q_j \quad \text{iff} \quad \inf_{\mathcal{U}\in\mathbb{U}(i)} \max_{u\in\mathcal{U}} \left(\mathbf{E}(u, \bigotimes^n p_j) - \mathbf{E}(u, \bigotimes^n q_j)\right) \geq 0 \tag{9}$$

for every $p_j, q_j \in \Delta(X_j)$ and $j \in [n]$. For each i in $[n]$, we pick an arbitrary u_i in $\bigcap \mathbb{U}(i)$. By Nash's Existence Theorem, there is a $(p_1^*, ..., p_n^*)$ in $\Delta(X_1) \times \cdots \times \Delta(X_n)$ such that

$$\mathbf{E}\left(u_i, \bigotimes^n (p_i^*, p_{-i}^*)\right) \geq \mathbf{E}\left(u_i, \bigotimes^n (p_i, p_{-i}^*)\right) \qquad \text{for every } p_i \in \Delta(X_i) \text{ and } i \in [n].$$

Since u_i belongs to every member of $\mathbb{U}(i)$, therefore,

$$\max_{u\in\mathcal{U}} \left(\mathbf{E}(u_i, \bigotimes^n p_j) - \mathbf{E}(u_i, \bigotimes^n q_j)\right) \geq 0$$

for each $\mathcal{U} \in \mathbb{U}(i)$ and $i \in [n]$. Given (9), it follows that (8) holds, and we are done. ■

Remark. The only other result on the existence of Nash equilibria with nontransitive preferences that we are aware of is the existence theorem of Fishburn and Rosenthal (1986). That result says that a strategic game with finitely many players and finite action spaces has a mixed strategy Nash equilibrium, provided that the preferences of each player (over mixed strategy profiles) is represented by an SSB utility (à la Fishburn (1982)). This theorem and Proposition 2 are not nested. Indeed, as we have noted earlier, an affine relation that is represented by an SSB utility must be of the expected utility form. Thus, the intersection of the Fishburn-Rosenthal theorem and Proposition 2 is precisely Nash's original existence theorem. □

4.5 Preferences over Monetary Lotteries

In this set of applications we concentrate on monetary lotteries (with compact support), and hence set $X := [0, 1]$. Our immediate objective is to characterize the structure of (stochastically) monotonic and/or risk averse preferences over such lotteries which need not be either transitive or complete. We then look at the structure of such preferences when they satisfy a stronger monotonicity condition with respect to (any sort of) stochastic dominance ordering. Finally, we put these findings in use by investigating if and when such preferences find portfolio diversification desirable.

4.5.1 Stochastic Monotonicity

Let us denote the **first-order stochastic dominance** relation on $\Delta[0,1]$ by $\geq_{\text{FSD}}$, and recall that $p \geq_{\text{FSD}} q$ iff $\mathbf{E}(u,p) \geq \mathbf{E}(u,q)$ for every increasing $u \in \mathbf{C}[0,1]$. In turn, we say that a binary relation $\mathbf{R}$ on $\Delta[0,1]$ is **stochastically monotonic** if $p \geq_{\text{FSD}} q$ implies $p \mathbf{R} q$ for any p and q in $\Delta[0,1]$.

Lemma 1. *Let $\mathbb{U}$ be a nonempty collection of nonempty compact and convex subsets of $\mathbf{C}[0,1]$, and let $\mathbf{R}$ be the binary relation on $\Delta[0,1]$ which satisfies (3) for every p and q in $\Delta[0,1]$. Then, $\mathbf{R}$ is stochastically monotonic if, and only if, each $\mathcal{U} \in \mathbb{U}$ contains an increasing function.*

Proof. The "if" part of the claim is straightforward. To prove its "only if" part, assume that $\mathbf{R}$ is stochastically monotonic. Let $\mathcal{V}$ stand for the set of all continuous and increasing real maps on $[0,1]$, and suppose $\mathcal{U} \cap \mathcal{V} = \emptyset$ for some $\mathcal{U} \in \mathbb{U}$. Since $\mathcal{U}$ is a nonempty compact and convex set, and $\mathcal{V}$ is a closed convex cone, in $\mathbf{C}[0,1]$, we can strongly separate $\mathcal{U}$ and $\mathcal{V}$ by a closed hyperplane in $\mathbf{C}[0,1]$, that is, there is a nonzero continuous linear functional L on $\mathbf{C}[0,1]$ such that $L(u) > 0 \geq L(v)$ for every $(u,v) \in \mathcal{U} \times \mathcal{V}$.[24] But, by the classical Riesz-Radon Representation Theorem and the Jordan Decomposition Theorem,

$$L(f) = \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu \quad \text{for every } f \in \mathbf{C}[0,1]$$

where μ and ν are two Borel measures on $[0,1]$. As L is nonzero, we have $\mu \neq \nu$ and $\min\{\mu(X), \nu(X)\} > 0$. Moreover, since the constant functions $\mathbf{1}_{[0,1]}$ and $-\mathbf{1}_{[0,1]}$ belong to $\mathcal{V}$, both $L(\mathbf{1}_{[0,1]})$ and $-L(\mathbf{1}_{[0,1]})$ are nonpositive, which implies that $\mu(X) = \nu(X) > 0$. Consequently, $p := \mu/\mu(X)$ and $q := \nu/\nu(X)$ are two lotteries on $[0,1]$ such that

$$\mathbf{E}(u,p) - \mathbf{E}(u,q) > 0 \geq \mathbf{E}(v,p) - \mathbf{E}(v,q) \quad \text{for every } (u,v) \in \mathcal{U} \times \mathcal{V}.$$

In view of the definition of $\mathbf{R}$, the first part of these inequalities implies that $q \mathbf{R} p$ does not hold. On the other hand, the second part of these inequalities entails $q \geq_{\text{FSD}} p$. As $\mathbf{R}$ is stochastically monotonic, then, $q \mathbf{R} p$, a contradiction. ■

Remark. We say that a binary relation $\mathbf{R}$ on $\Delta[0,1]$ is **stochastically strictly monotonic** if $p >_{\text{FSD}} q$ implies $p \mathbf{R}^{>} q$ for any p and q in $\Delta[0,1]$. It is an easy exercise to show that, under the conditions of Lemma 1, $\mathbf{R}$ is stochastically strictly monotonic iff (a) each $\mathcal{U} \in \mathbb{U}$ contains an increasing function; and (b) for every $x, y \in [0,1]$ with $x > y$, there is a $\mathcal{U} \in \mathbb{U}$ such that $u(x) > u(y)$ for each $u \in \mathcal{U}$. □

Combining Lemma 1 with Theorem 4 yields the characterization we are after.

Proposition 3. *Let $\mathbf{R}$ be a binary relation on $\Delta[0,1]$. Then, $\mathbf{R}$ is a stochastically monotonic and continuous affine relation on $\Delta[0,1]$ if, and only if, there is a compact collection $\mathbb{U}$ of nonempty compact and convex sets of continuous real maps on $[0,1]$ such that (i) $\mathbb{U}$ is a coalitional minmax expected utility for $\mathbf{R}$; and (ii) each $\mathcal{U} \in \mathbb{U}$ contains an increasing function.*

[24] See, for instance, Theorem 5.58 in Aliprantis and Border (2006).

Remark. This result extends the characterization provided in Proposition 2 of Dubra, Maccheroni and Ok (2004) for (possibly incomplete) stochastically monotonic affine preorders by dropping the transitivity assumption. □

4.5.2 Risk Aversion

Let us denote the **mean** of any lottery p on $[0,1]$ by $e(p)$, that is, $e(p) := \mathbf{E}(\mathrm{id}_{[0,1]}, p)$ where $\mathrm{id}_{[0,1]}$ is the identity map on $[0,1]$. A binary relation $\mathbf{R}$ on $\Delta[0,1]$ is said to be **weakly risk averse** if $\delta_{e(p)}\ \mathbf{R}\ p$ for every $p \in \Delta[0,1]$. There are useful characterizations of weakly risk averse binary relations that admit an expected utility representation, but it is well-known that weak risk aversion becomes an unduly weak property the moment we leave the classical expected utility paradigm (cf. Cohen (1995)). Instead, the behavioral notion of risk aversion in nonexpected utility theory is often modeled by what is called "strong risk aversion," which corresponds to the idea that any mean-preserving spread of a lottery makes that lottery less desirable. There is no difference between weak and strong risk aversion in the context of expected utility theory, but the latter is amenable to analysis in the case of more general preference relations on risky prospects. The present framework is no exception to this.

We say that a binary relation $\mathbf{R}$ on $\Delta[0,1]$ is **strongly risk averse** if $p\ \mathbf{R}\ q$ for any p and q in $\Delta[0,1]$ such that $e(p) = e(q)$ and $\mathbf{E}(u,p) \geq \mathbf{E}(u,q)$ for every concave $u \in \mathbf{C}[0,1]$. In the context of preferences that admit a coalitional minmax expected utility representation, we have the following analogue of Lemma 1.

Lemma 2. *Let $\mathbb{U}$ be a nonempty collection of nonempty compact and convex subsets of $\mathbf{C}[0,1]$, and let $\mathbf{R}$ be the binary relation on $\Delta[0,1]$ which satisfies (3) for every p and q in $\Delta[0,1]$. Then, $\mathbf{R}$ is strongly risk averse if, and only if, each $\mathcal{U} \in \mathbb{U}$ contains a concave function.*

Proof. The "if" part of the assertion is immediate from (3) and the definition of strong risk aversion. To prove the "only if" part, let $\mathcal{V}$ stand for the set of all continuous and concave real maps on $[0,1]$, and suppose $\mathcal{U} \cap \mathcal{V} = \emptyset$ for some $\mathcal{U} \in \mathbb{U}$. Proceeding exactly as in the proof of Lemma 1, then, we find two lotteries p and q on $[0,1]$ such that

$$\mathbf{E}(u,p) - \mathbf{E}(u,q) > 0 \geq \mathbf{E}(v,p) - \mathbf{E}(v,q) \quad \text{for every } (u,v) \in \mathcal{U} \times \mathcal{V}.$$

Evaluating the second part of these inequalities with $\mathrm{id}_{[0,1]}$ and $-\mathrm{id}_{[0,1]}$, we see that $e(p) = e(q)$, so it follows from this part that $q\ \mathbf{R}\ p$ would hold if $\mathbf{R}$ were strongly risk averse. Yet (3) and the first part of these inequalities imply that $q\ \mathbf{R}\ p$ does not hold. ■

Once again, combining Lemma 2 with Theorem 4 yields the characterization we are after.

Proposition 4. *Let $\mathbf{R}$ be a binary relation on $\Delta[0,1]$. Then, $\mathbf{R}$ is a strongly risk averse and continuous affine relation on $\Delta[0,1]$ if, and only if, there is a compact collection $\mathbb{U}$ of nonempty compact and convex sets of continuous real maps on $[0,1]$ such that* (i) *$\mathbb{U}$ is a coalitional minmax expected utility for $\mathbf{R}$; and* (ii) *each $\mathcal{U} \in \mathbb{U}$ contains a concave function.*

Remark. It is plain that every strongly risk averse binary relation $\mathbf{R}$ on $\Delta[0,1]$ is weakly risk averse. We can use the representation obtained in Theorem 5 to show that the converse of this is false even when $\mathbf{R}$ is continuous, quasitransitive and affine. Indeed, we can easily distort the identity function to obtain two increasing, continuous and nonconcave real maps on $[0,1]$ such that the convex hull $\mathcal{U}$ of $\{u,v\}$ does not contain any concave functions while for each $p \in \Delta[0,1]$, either $u(e(p)) = e(p) \geq \mathbf{E}(u,p)$ or $v(e(p)) = e(p) \geq \mathbf{E}(v,p)$.[25] Now put $\mathbb{U} := \{\mathcal{U}\}$ and define $\mathbf{R}$ by (3). Then, by Theorem 5, $\mathbf{R}$ is continuous, strictly convex and affine (and hence quasitransitive). On the other hand, while it is plain that it is weakly risk averse, Lemma 2 says that $\mathbf{R}$ is not strongly risk averse. □

4.5.3 Transitivity with respect to Stochastic Dominance

Suppose that $\geq$ is a partial order on $\Delta[0,1]$ which we take as some sort of a dominance relation on the space of monetrary lotteries. The interpretation is that if $p \geq q$, we think of p is a better lottery than q in some objective sense. Now consider an individual whose preferences are modeled by means of a binary relation $\mathbf{R}$ on $\Delta[0,1]$, and suppose that this individual recognizes $\geq$ indeed as an unambigous, albeit partial, ordering of the lotteries. Then, clearly, $p \geq q$ would imply $p \ \mathbf{R} \ q$ for any two lotteries p and q on $[0,1]$. But, given the interpretation of $\geq$, it makes sense to ask for the consistency of $\mathbf{R}$ with $\geq$ in a stronger sense. Suppose we know that $p \geq q$ and $q \ \mathbf{R} \ r$. In this case, p is better than q in some "obvious" sense, while q is preferred to r by this individual. Even if her preferences may exhibit cycles, it is not unreasonable to limit the structure of such cycles, and presume that we would have $p \ \mathbf{R} \ r$ in this case. When this holds for every p, q and r in $[0,1]$, we say that $\mathbf{R}$ is **transitive with respect to** $\geq$.

The suitability of "transitivity with respect to $\geq$" as a behavioral property depends on the kind of application that one is interested in, and of course, in the exact nature of $\geq$. One case that is of obvious interest is when $\geq$ is the first-order stochastic dominance ordering. When $\mathbf{R}$ is transitive with respect to $\geq_{\text{FSD}}$, we say that it is **FSD-transitive**. Another case of interest is when $\geq$ is the second-order stochastic dominance ordering. (We denote the **second-order stochastic dominance** relation on $\Delta[0,1]$ by $\geq_{\text{SSD}}$, and recall that $p \geq_{\text{SSD}} q$ iff $\mathbf{E}(u,p) \geq \mathbf{E}(u,q)$ for every increasing and concave $u \in \mathbf{C}[0,1]$.) When $\mathbf{R}$ is transitive with respect to $\geq_{\text{SSD}}$, we say that it is **SSD-transitive.**

The following result characterizes the structure of all FSD- and SSD-transitive continuous affine relations on $\Delta[0,1]$.

Proposition 5. *Let $\mathbf{R}$ be a binary relation on $\Delta[0,1]$. Then, $\mathbf{R}$ is an FSD-transitive (SSD-transitive) and continuous affine relation on $\Delta[0,1]$ if, and only if, there is a compact collection of nonempty compact and convex sets of continuous, increasing (and concave) real maps on $[0,1]$ which is a coalitional minmax expected utility for $\mathbf{R}$.*

As we shall see in the next section, this proposition makes it quite easy to work with FSD- and SSD-transitive continuous affine relations on $\Delta[0,1]$. In passing, we note that we focus on these two partial transitivity conditions here only for concreteness. The proposition extends easily to cover the case of continuous and affine relations on $\Delta[0,1]$ that are transitive with respect to kth order stochactic dominance ordering (for any positive integer k).

[25] To give a concrete example, let $u(t) := t$ for each $t \in [0, \frac{1}{2})$, $u(t) := \frac{1}{2}$ for each $t \in [\frac{1}{2}, \frac{3}{4})$, and $u(t) := 2t-1$ for each $t \in (\frac{3}{4}, 1]$. Then set $v(t) := 0$ for each $t \in [0, \frac{1}{4})$, $v(t) := 2t - \frac{1}{2}$ for each $t \in [\frac{1}{4}, \frac{1}{2})$, and $v(t) := t$ for each $t \in (\frac{1}{2}, 1]$.

4.5.4 Portfolio Diversification with Non-transitive Preferences

By an **asset**, we mean a random variable on the measure space $([0,1], \mathcal{B}, \ell)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ and ℓ is the Lebesgue measure on $[0,1]$. We denote the Borel probability measure that an asset x induces on $[0,1]$, that is, the *distribution* of x, by p_x. (That is, $p_x := \ell \circ x^{-1}$.) Given any binary relation $\mathbf{R}$ on $\Delta[0,1]$, which models the preferences of an investor over lotteries on $[0,1]$, we define the investor's preferences over the set $\mathcal{A}$ of all assets by means of the binary relation $\mathbf{R}_*$ on $\mathcal{A}$ with

$$x \ \mathbf{R}_* \ y \quad \text{iff} \quad p_x \ \mathbf{R} \ p_y$$

for every $x, y \in \mathcal{A}$. Adopting the formulation of Dekel (1989), then, we say that a binary relation $\mathbf{R}$ on $\Delta[0,1]$ **exhibits preference for two-asset portfolio diversification** if

$$x \ \mathbf{R}_*^{=} \ y \quad \text{implies} \quad \lambda x + (1-\lambda) y \ \mathbf{R}_* \ \{x, y\}$$

for every $x, y \in \mathcal{A}$ and $\lambda \in [0,1]$. If $\mathbf{R}$ admits an expected utility representation, and is risk averse, then a straightforward application of Jensen's Inequality shows that it exhibits preference for two-asset portfolio diversification. We now show that this fact remains true in the context of a large class of risk averse preference relations that need not be either complete or transitive.

Proposition 6. *Let* $\mathbf{R}$ *be a continuous and SSD-transitive affine relation on* $\Delta[0,1]$. *Then,* $\mathbf{R}$ *exhibits preference for two-asset portfolio diversification.*

Proof. By Proposition 5, there is a coalitional minmax expected utility $\mathbb{U}$ for $\mathbf{R}$ such that every member of $\bigcup \mathbb{U}$ is increasing and concave. Take any $x, y \in \mathcal{A}$ with $x \ \mathbf{R}_*^{=} \ y$. Then, for every $\mathcal{U} \in \mathbb{U}$ there is a $u_{\mathcal{U}} \in \mathcal{U}$ such that $\mathsf{E}(u_{\mathcal{U}}, p_y) \geq \mathsf{E}(u_{\mathcal{U}}, p_x)$. Then, for each $\mathcal{U} \in \mathbb{U}$, using the concavity of $u_{\mathcal{U}}$ and the Change of Variables Formula yields

$$\begin{aligned} \mathsf{E}(u_{\mathcal{U}}, p_{\lambda x + (1-\lambda) y}) &= \mathsf{E}(u_{\mathcal{U}} \circ (\lambda x + (1-\lambda) y), \ell) \\ &\geq \lambda \mathsf{E}\left(u_{\mathcal{U}} \circ x, \ell\right) + (1-\lambda) \mathsf{E}\left(u_{\mathcal{U}} \circ y, \ell\right) \\ &= \lambda \mathsf{E}(u_{\mathcal{U}}, p_x) + (1-\lambda) \mathsf{E}(u_{\mathcal{U}}, p_y) \\ &\geq \mathsf{E}(u_{\mathcal{U}}, p_x), \end{aligned}$$

that is, $\lambda x + (1-\lambda) y \ \mathbf{R}_* \ x$, for any $\lambda \in [0,1]$. Replacing the roles of x and y in this argument completes the proof. ■

In the case of expected utility theory, restricting attention to two-asset portfolios is without loss of generality. To wit, let us say that a binary relation $\mathbf{R}$ on $\Delta[0,1]$ **exhibits preference for n-asset portfolio diversification** if

$$x^i \ \mathbf{R}_*^{=} \ x^j \text{ for each } i, j \in [n] \quad \text{implies} \quad \lambda_1 x^1 + \cdots + \lambda_n x^n \ \mathbf{R}_* \ \{x^1, ..., x^n\}$$

for every $x^1, ..., x^n \in \mathcal{A}$ and $\lambda_1, ..., \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$. A classical result of expected utility theory, which goes back to Tobin (1958), says that if $\mathbf{R}$ is risk averse, and has an expected utility representation, then it exhibits preference for n-asset portfolio diversification for any positive integer n. At first glance, it appears that transitivity hypothesis is not of the essence for the validity of this fact. Indeed, Proposition 6 supports this intuition. Surprisingly, however, this result does not extend beyond two-asset portfolios. In fact, even in the context of risk averse justifiable preferences, preference for portfolio diversification does not hold in general. As our final application, we prove this by using Theorem 5.

Observation. *A continuous and SSD-transitive affine relation on* $\Delta[0,1]$ *need not exhibit preference for three-asset portfolio diversification.* To see this, consider the real concave maps u and v on $[0,1]$ defined as:

$$u(t) := \begin{cases} 4t, & \text{if } 0 \leq t < \frac{1}{8} \\ \frac{4}{7}t + \frac{3}{7}, & \text{if } \frac{1}{8} \leq t \leq 1 \end{cases} \qquad \text{and} \qquad v(t) := t.$$

Let $\mathbf{R}$ be the binary relation on $\Delta[0,1]$ for which $\{\text{co}\{u,v\}\}$ is a coalitional minmax expected utility. Then, by Theorem 5 and Proposition 5, $\mathbf{R}$ is a continuous, complete, strictly convex and SSD-transitive (hence strongly risk averse) affine relation on $\Delta[0,1]$. (Moreover, this relation is strictly monotonic with respect to first-order stochastic dominance in the sense that $p \geq_{\text{FSD}} q$ implies $p\ \mathbf{R}^{>}\ q$ for any distinct p and q in $\Delta[0,1]$.) However, $\mathbf{R}$ does *not* exhibit preference for three-asset portfolio diversification. To see this, consider the assets $x := \frac{1}{8}\mathbf{1}_{[0,1]}$, $y := \frac{1}{2}\mathbf{1}_{[0,5/8)}$, and $z := \mathbf{1}_{[0,3/8)}$. The distributions of these assets are $p_x := \delta_{1/8}$, $p_y := \frac{5}{8}\delta_{1/2} + \frac{3}{8}\delta_0$ and $p_z := \frac{3}{8}\delta_1 + \frac{5}{8}\delta_0$, respectively. Easy computations show that

$$\mathbf{E}(u,p_x) > \mathbf{E}(u,p_y) > \mathbf{E}(u,p_z) \quad \text{and} \quad \mathbf{E}(v,p_z) > \mathbf{E}(v,p_y) > \mathbf{E}(v,p_x),$$

which means that $p_x\ \mathbf{R}^{=}\ p_y\ \mathbf{R}^{=}\ p_z\ \mathbf{R}^{=}\ p_x$. Now let $l := \frac{1}{3}x + \frac{2}{3}z$. Then, $p_l := \frac{3}{8}\delta_{17/24} + \frac{5}{8}\delta_{1/24}$, and another easy computation yields

$$\mathbf{E}(u,p_y) > \mathbf{E}(u,p_l) \quad \text{and} \quad \mathbf{E}(v,p_y) > \mathbf{E}(v,p_l),$$

which means $p_y\ \mathbf{R}^{>}\ p_l$. But then, by continuity, we can choose a small enough $\varepsilon > 0$ so that $p_y\ \mathbf{R}^{>}\ p_{\varepsilon l+(1-\varepsilon)y}$, that is, $y\ \mathbf{R}_*^{>}\ \varepsilon l + (1-\varepsilon)y$. As $\varepsilon l + (1-\varepsilon)y$ is a convex combination of x, y and z, our assertion is proved. □

5 Open Problems

We conclude our exposition by putting on record some of the open problems that arise from the present work.

O1. Theorems 5 and 6, respectively, characterize justifiable and acyclic continuous affine relations, albeit under the completeness hypothesis. The proofs of these results utilize this hypothesis in a crucial way, and it remains an open problem to find what effect dropping completeness assumption would have on these results. (At present, we do not even have a conjecture on this matter.) As Theorems 5 and 6 have proved extremely useful in the applications of Section 4, this appears to be a worthwhile task.

O2. Let $\mathbf{R}$ be an affine preorder, or a complete affine preorder, on $\Delta(X)$, where X is, say, a separable (or compact) metric space. We know from Theorem 1 that there is a coalitional minmax expected utility $\mathbb{U}$ for $\mathbf{R}$. We do not know, however, what exactly is the structure of $\mathbb{U}$. (Put another way, we do not know what sort of a coalitional representation we would get if we added the transitivity hypothesis to Theorems 1 and/or 2. Once again, we do not have a conjecture on this matter.) See Appendix B for more on this problem.

O3. Dubra, Maccheroni and Ok (2004) report a rather nice uniqueness result for their expected multi-utility representation (which generalizes the notion of "unique up to positive affine transformations.") A similar uniqueness result is valid for our characterization of justifiable preferences in Theorem 5. To put this precisely, let X be a compact metric space, and let $\mathcal{K}(X)$ be the collection of all nonempty compact and convex subsets of $\mathbf{C}(X)$. Define $\Lambda(\mathcal{U}) := \{au + b : a > 0,\ b \in \mathbb{R} \text{ and } u \in \mathcal{U}\}$ for any $\mathcal{U} \in \mathcal{K}(X)$. Theorem 5 says that for every continuous, complete, and strictly convex affine relation $\mathbf{R}$ on $\Delta(X)$, there is a $\mathcal{V}$ in $\mathcal{K}(X)$ such that $\{\mathcal{V}\}$ is a coalitional minmax expected utility for $\mathbf{R}$. Then, $\{\mathcal{U}\}$ is a coalitional minmax expected utility for $\mathbf{R}$ for some other $\mathcal{U}$ in $\mathcal{K}(X)$ iff $\Lambda(\mathcal{U}) = \Lambda(\mathcal{V})$. (We leave the proof as an exercise.) Obtaining similar uniqueness results for Theorems 1-4 turn out to be significantly more challenging. This task remains an open problem.

O4. We have carried out our analysis in this paper exclusively in the context of risk, taking the von Neumann-Morgenstern expected utility theorem as our starting point. It is a natural problem to extend this analysis to the context of uncertainty, where one would instead take the Anscombe-Aumann expected utility theorem as the starting point. Indeed, Lehrer and Teper (2011) have provided the formulation of justifiable preferences in this setup. Similarly, Nau (2006), Galaabaatar and Karni (2013) and Riella (2015) prove versions of the Anscombe-Aumann Theorem without the completeness hypothesis (albeit strengthening the monotonicity axiom and with finite state and prize spaces). Nothing is known at present about the structure of affine and monotonic (and state independent) preferences over (finite) acts (on a finite state space) which fail to satisfy continuity, completeness and/or transitivity. The theorems of the present paper are likely to be instrumental in the study of this problem (for they settle the matter for constant acts), but needless to say, much more work remains to be done.

APPENDIX A:

This appendix contains the proofs of our main results. We begin, however, with a brief review of some of the mathematical notions that we will use below.

A.1: Preliminaries

Convex Cones. A nonempty subset C of a (real) linear space Y is said to be a *cone* if $\lambda C \subseteq C$ for every $\lambda \geq 0$. If, in addition, $C + C \subseteq C$, then C is a convex set, and we then say that it is a *convex cone.*[26] The intersection of an arbitrary nonempty collection of convex cones in Y is a convex cone in Y, and the union of an arbitrary chain of convex cones in Y is a convex cone in Y. Thanks to the former property, for any nonempty set S in Y, there is a (unique) smallest convex cone that contains S, which we denote by $\text{cone}(S)$. It is easily checked that $\text{cone}(S)$ equals the set of all elements of the form $\sum^n \lambda_i y_i$ where n is any positive integer, $\lambda_1, ..., \lambda_n$ are nonnegative real numbers, and $y_1, ..., y_n \in S$. In particular, $\text{cone}\{y\} = \{\lambda y : \lambda \geq 0\}$ for any $y \in Y$, and more generally, $\text{cone}(S) = \{\lambda y : \lambda \geq 0 \text{ and } y \in S\}$ for any nonempty convex subset S of Y. If S is a nonempty finite set and $C = \text{cone}(S)$, we say that C is *finitely generated.* It is well-known that every finitely generated convex cone in a Hausdorff topological linear space is closed.

[26] As is standard, by $\alpha A + \beta B$ we mean the set $\{\alpha a + \beta b : (a, b) \in A \times B\}$ for any real numbers α and β, and any subsets A and B of the linear space Y.

We will need the following simple observation about convex cones in the sequel.

Lemma A.1. Let C be a convex cone in a linear space Y and take any $x \in Y \backslash C$. Then, there is a convex cone C_x in Y such that (i) $C \subseteq C_x$; (ii) $x \in Y \backslash C_x$; and (iii) $C_x \cup -C_x = Y$.

Proof. Let $\mathcal{D}$ stand for the set of all convex cones in Y such that $C \subseteq D$ and $x \in Y \backslash D$. As it contains C, this set is not empty. As the union of any chain of convex cones in Y is a convex cone in Y, we may apply Zorn's Lemma to find a maximal element, say, C_x, of $\mathcal{D}$ (with respect to set containment). As it belongs to $\mathcal{D}$, the cone C_x satisfies (i) and (ii). Next, to derive a contradiction, suppose $Y \backslash (C_x \cup -C_x)$ is not empty. Then, there is a $y \in Y$ such that neither y nor $-y$ belongs to C_x. Define

$$D_1 := \text{ cone}(C_x \cup \{y\}) \quad \text{ and } \quad D_2 := \text{ cone}(C_x \cup \{-y\}).$$

By the maximality of C_x, both D_1 and D_2 must contain x. It follows that $x = z_1 + \alpha y$ and $x = z_2 + \beta(-y)$ for some $z_1, z_2 \in C_x$ and $\alpha, \beta > 0$. But then $z_2 - z_1 = (\alpha + \beta)y$, and hence $y = \frac{1}{\alpha+\beta}(z_2 - z_1)$, which implies

$$x = z_1 + \alpha y = z_1 + \left(\frac{\alpha}{\alpha+\beta}\right)(z_2 - z_1) = \frac{\alpha}{\alpha+\beta} z_2 + \left(1 - \frac{\alpha}{\alpha+\beta}\right) z_1.$$

As C_x is convex, this implies that $x \in C_x$, a contradiction. Conclusion: C_x satisfies (iii). ■

ca(X). In what follows ca(X) stands for the linear space of signed finite Borel measures on X. We view this space as a normed linear space relative to the total variation norm $\|\cdot\|_{\text{TV}}$, where $\|\mu\|_{\text{TV}} := |\mu|(X)$, and denote its closed unit ball by B, that is, $B := \{\mu \in \text{ca}(X) : \|\mu\|_{\text{TV}} \leq 1\}$. Let us now assume that X is a compact metric space. Under this hypothesis, the normed linear space ca(X) is isometrically isomorphic to $\mathbf{C}(X)^*$. (This is the Radon-Riesz Representation Theorem.) We use this duality to topologize ca(X) with the weak*-topology. Those subsets of ca(X) that are open (closed) with respect to this topology are said to be *w^*-open* (*w^*-closed*). Note that a net (μ_α) in ca(X) converges to a signed finite Borel measure μ on X relative to the weak*-topology iff

$$\int_X f d\mu_\alpha \to \int_X f d\mu \quad \text{ for every } f \in \mathbf{C}(X).$$

We will also need to use another topology on ca(X), namely, the *bounded weak* topology* on ca(X). By definition, this topology is the strongest (i.e. finest) topology on ca(X) which agrees with the weak* topology on every bounded subset of ca(X). Put differently, a set $O \subseteq \text{ca}(X)$ is open (closed) with respect to this topology – we say in this case that O is *bw^*-open* (*bw^*-closed*) – iff $O \cap \lambda B$ is w^*-open (w^*-closed) in λB for every $\lambda > 0$. More generally, for any nonempty subset $\mathcal{S}$ of ca(X), a set $O \subseteq \mathcal{S}$ is bw^*-open (bw^*-closed) in $\mathcal{S}$ iff $O \cap \lambda B$ is w^*-open (w^*-closed) in $\mathcal{S} \cap \lambda B$ for every $\lambda > 0$.

It is known that topologizing ca(X) with the bounded weak*-topology is the same as topologizing this linear space with the topology of uniform convergence on compact subsets of $\mathbf{C}(X)$. (Relative to the latter topology, a net (μ_α) in ca(X) converges to a $\mu \in \text{ca}(X)$ iff

$$\sup_{f \in F} \left| \int_X f d\mu_\alpha - \int_X f d\mu \right| \to 0 \quad \text{ for any nonempty compact } F \subseteq \mathbf{C}(X).)$$

As such, ca(X) is an Hausdorff locally convex topological linear space relative to the bounded weak*-topology. Moreover, a linear functional on ca(X) is continuous with respect to the weak* topology iff it is continuous with respect to the bounded weak* topology. This fact leads to the famous Krein-Šmulian Theorem: A convex subset of ca(X) is w^*-closed iff it is bw^*-closed.[27]

$\mathcal{S}(X)$. In what follows, we put

$$\mathcal{S}(X) := \{\mu \in \text{ca}(X) : \mu(X) = 0\}$$

which is a linear subspace of ca(X). A straightforward application of the Jordan Decomposition Theorem shows that this space is equal to the linear span of $\Delta(X) - \Delta(X)$. When ca(X) is given a particular topology,

[27] All of the facts we have stated in this paragraph are standard, but none is trivial. For proofs of them, see, for instance, Section 8.8 of Aliprantis and Tourky (2007).

we consider $\mathcal{S}(X)$ as a topological space itself by endowing it with the subspace topology. In particular, when $\mathrm{ca}(X)$ is endowed with either the weak* or the bounded weak* topology, $\mathcal{S}(X)$ becomes a Hausdorff locally convex topological linear space.

Aumann Cones. An extremely useful idea in the study of affine relations on a mixture space is to think of such relations as cones in a suitable linear space. This idea was introduced by Aumann (1962) in the special case of affine preorders, and since then became a common method of analysis in expected utility theory.

Let X be a metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$. We define

$$C(\mathbf{R}) := \{\lambda(r - s) : \lambda \geq 0 \text{ and } r \ \mathbf{R}\ s\}$$

and

$$C_0(\mathbf{R}) := \{\lambda(r - s) : \lambda > 0 \text{ and } r \ \mathbf{R}\ s\},$$

notations that we will use throughout this appendix. Clearly, both $C(\mathbf{R})$ and $C_0(\mathbf{R}) \cup \{\mathbf{0}\}$, where $\mathbf{0}$ stands for the zero Borel measure on X, are cones in $\mathrm{ca}(X)$. The following two observations are building blocks of every result that we report in this paper.

Lemma A.2. Let X be a metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$ which satisfies the Independence Axiom.

(a) If $\mathbf{R}$ is irreflexive, then $p \ \mathbf{R}\ q$ iff $p - q \in C_0(\mathbf{R})$, for any $p, q \in \Delta(X)$.
(b) If $\mathbf{R}$ is irreflexive, then $C_0(\mathbf{R}) \cup \{\mathbf{0}\}$ is a convex cone in $\mathcal{S}(X)$ if, and only if, $\mathbf{R}$ is transitive.
(c) If $\mathbf{R}$ is reflexive, then $p \ \mathbf{R}\ q$ iff $p - q \in C(\mathbf{R})$, for any $p, q \in \Delta(X)$.
(d) If $\mathbf{R}$ is reflexive, then $C(\mathbf{R})$ is a convex cone in $\mathcal{S}(X)$ if, and only if, $\mathbf{R}$ is transitive.

Proof. Suppose $\mathbf{R}$ is irreflexive, and take any $p, q \in \Delta(X)$. Obviously, $p \ \mathbf{R}\ q$ implies $p - q \in C_0(\mathbf{R})$. Conversely, suppose $p - q = \lambda(r - s)$ for some $\lambda > 0$ and $r \ \mathbf{R}\ s$. Then, by affinity of $\mathbf{R}$,

$$\frac{1}{1+\lambda}p + \frac{\lambda}{1+\lambda}s = \frac{1}{1+\lambda}q + \frac{\lambda}{1+\lambda}r \ \mathbf{R}\ \frac{1}{1+\lambda}q + \frac{\lambda}{1+\lambda}s,$$

which, again by affinity of $\mathbf{R}$, yields $p \ \mathbf{R}\ q$, and part (a) is established. Moreover, if $\mathbf{R}$ is reflexive, we can allow λ to take value 0 in this argument, so part (c) is established as well. To prove (d), assume that $\mathbf{R}$ is reflexive, and suppose first that $C(\mathbf{R})$ is a convex cone in $\mathcal{S}(X)$. Then, for any $p, q, r \in \Delta(X)$ with $p \ \mathbf{R}\ q \ \mathbf{R}\ r$, part (c) entails that $p - r = (p - q) + (q - r) \in C(\mathbf{R}) + C(\mathbf{R}) \subseteq C(\mathbf{R})$, and hence, again by part (c), we find $p \ \mathbf{R}\ r$, as we sought. Conversely, suppose $\mathbf{R}$ is a preorder on $\Delta(X)$, and take any μ_1 and μ_2 in $C(\mathbf{R})$. We assume that neither of these signed measures is the zero measure, for otherwise $\mu_1 + \mu_2$ is trivially in $C(\mathbf{R})$. Then, for each $i \in \{1, 2\}$, there are $\lambda_i > 0$ and $(r_i, s_i) \in \mathbf{R}$ such that $\mu_i = \lambda_i(r_i - s_i)$. Let $\lambda := \frac{\lambda_1}{\lambda_1+\lambda_2}$, and note that

$$\lambda r_1 + (1-\lambda)r_2 \ \mathbf{R}\ \lambda r_1 + (1-\lambda)s_2 \ \mathbf{R}\ \lambda s_1 + (1-\lambda)s_2$$

because $\mathbf{R}$ satisfies the Independence Axiom. As $\mathbf{R}$ is transitive, therefore, $\lambda r_1 + (1-\lambda)r_2 \ \mathbf{R}\ \lambda s_1 + (1-\lambda)s_2$, and hence, by part (c), $\lambda(r_1 - s_1) + (1-\lambda)(r_2 - s_2)$ belongs to $C(\mathbf{R})$. Thus:

$$\mu_1 + \mu_2 = (\lambda_1 + \lambda_2)\left(\lambda(r_1 - s_1) + (1-\lambda)(r_2 - s_2)\right) \in (\lambda_1 + \lambda_2)C(\mathbf{R}) \subseteq C(\mathbf{R}).$$

We thus conclude that $C(\mathbf{R}) + C(\mathbf{R}) \subseteq C(\mathbf{R})$, and part (d) is proved. It remains to prove part (b), but this case is deduced readily from part (d) by using the fact that $C(\mathbf{S}) = C_0(\mathbf{R}) \cup \{\mathbf{0}\}$, where $\mathbf{S} = \mathbf{R} \cup \{(p, p) : p \in \Delta(X)\}$. ■

Lemma A.3. Let X be a compact metric space and $\mathbf{R}$ a binary relation on $\Delta(X)$ which satisfies the Independence Axiom.

(a) If $\mathbf{R}$ is w^*-open and irreflexive, then $C_0(\mathbf{R})$ is bw^*-open in $\mathcal{S}(X)$.
(b) If $\mathbf{R}$ is w^*-closed and reflexive, then $C(\mathbf{R})$ is bw^*-closed in $\mathcal{S}(X)$.[28]

Proof. Assume that $\mathbf{R}$ is w^*-open. To prove part (a), we need the following fact.

[28] This observation relates closely to the main lemma of Dubra, Maccheroni and Ok (2004) who prove that if $\mathbf{R}$ is a w^*-closed affine preorder, then $C(\mathbf{R})$ is w^*-closed in $\mathcal{S}(X)$.

Claim A.3.1. $C_0(\mathbf{R}) \cap \lambda B$ is w^*-open in $\mathcal{S}(X) \cap \lambda B$, for every $\lambda > 0$.

Proof of Claim A.3.1. Take any $\lambda > 0$, and put $\mathcal{K} := \mathcal{S}(X) \cap \lambda B$ to simplify the notation. Let (μ_m) be any sequence in $\mathcal{K} \backslash C_0(\mathbf{R})$ which converges to some $\mu \in \mathcal{K}$. We wish to show that $\mu \in \mathcal{K} \backslash C_0(\mathbf{R})$. If $\mu_m = \mathbf{0}$ for infinitely many m, then $\mu = \mathbf{0}$, and we are done. Otherwise, we may assume without loss of generality that $\mu_m \neq \mathbf{0}$ for every m. By the Jordan Decomposition Theorem, for each $m \in \mathbb{N}$ there exists a $(\theta_m, p_m, q_m) \in \mathbb{R}_{++} \times \Delta(X) \times \Delta(X)$ such that $\mu_m = \theta_m(p_m - q_m)$ and $\|p_m - q_m\|_{\mathrm{TV}} = 2$. Since $\|\mu_m\|_{\mathrm{TV}} \leq \lambda$, this implies that $\sup \theta_m \leq \frac{\lambda}{2}$, that is, (θ_m) is bounded. As compactness of X ensures that $\Delta(X) \times \Delta(X)$ is weak*-compact in ca$(X)\times$ ca(X), therefore, there is a strictly increasing sequence (m_k) in $\mathbb{N}$ such that (θ_{m_k}) and (p_{m_k}, q_{m_k}) converge in $\mathbb{R}_+$ and in $\Delta(X) \times \Delta(X)$, respectively. Of course, where $\theta := \lim \theta_{m_k}$, $p := \lim p_{m_k}$ and $q := \lim q_{m_k}$, we have $\mu = \theta(p - q)$. But $p_{m_k} - q_{m_k}$ does not belong to $C_0(\mathbf{R})$, so by part (b) of Lemma A.2, (p_{m_k}, q_{m_k}) is a sequence in $(\Delta(X) \times \Delta(X)) \backslash \mathbf{R}$. As this is w^*-closed by hypothesis, it follows that it contains (p, q) as well, that is, $p - q$ does not belong to $C_0(\mathbf{R})$, as we sought. Conclusion: $C_0(\mathbf{R})$ is sequentially weak*-open in $\mathcal{K}$. As it is well-known that weak* topology on λB is metrizable, we are done. ‖

Now, fix an arbitrary $\lambda > 0$, and write again $\mathcal{K} := \mathcal{S}(X) \cap \lambda B$. We know from Claim A.3.1 that $C_0(\mathbf{R}) \cap \lambda B$ is w^*-open in $\mathcal{K}$, that is, $\mathcal{K} \backslash (C_0(\mathbf{R}) \cap \lambda B)$ is w^*-closed in $\mathcal{K}$. Since

$$(\mathcal{S}(X) \backslash C_0(\mathbf{R})) \cap \lambda B = \mathcal{K} \backslash (C_0(\mathbf{R}) \cap \lambda B),$$

we find that $(\mathcal{S}(X) \backslash C_0(\mathbf{R})) \cap \lambda B$ is w^*-closed in $\mathcal{K}$. Since, being the intersection of two w^*-closed sets, $\mathcal{K}$ is w^*-closed in ca(X), we may conclude that $(\mathcal{S}(X) \backslash C_0(\mathbf{R})) \cap \lambda B$ is w^*-closed in ca(X). As this is true for any $\lambda > 0$, therefore, $\mathcal{S}(X) \backslash C_0(\mathbf{R})$ is bw^*-closed in ca(X), which obviously implies that $\mathcal{S}(X) \backslash C_0(\mathbf{R})$ is bw^*-closed in $\mathcal{S}(X)$, thereby proving part (a) of the lemma.

To prove part (b), assume that $\mathbf{R}$ is w^*-closed and reflexive, and define $\mathbf{S} := (\Delta(X) \times \Delta(X)) \backslash \mathbf{R}$. Then, $\mathbf{S}$ is an irreflexive and w^*-open binary relation on $\Delta(X)$ which satisfies the Independence Axiom. Moreover, using Lemma A.2, one can easily check that $C(\mathbf{R}) = \mathcal{S}(X) \backslash C_0(\mathbf{S})$. Applying part (a) to $\mathbf{S}$, therefore, we find that $C(\mathbf{R})$ is bw^*-closed in $\mathcal{S}(X)$. ■

Affine Duality. We will use the following duality theorem frequently in the subsequent arguments.

The Affine Representation Lemma. *Let X be a separable metric space, and F a continuous and affine real map on $\Delta(X)$. Then, there is a continuous and bounded $u : X \to \mathbb{R}$ such that*

$$F(p) = \int_X u dp \quad \text{for every } p \in \Delta(X).$$

Although we do not know a concrete reference to give for it, it is safe to say that this theorem is well-known in the decision theory folklore. We thus omit its proof here, but note that the argument would proceed by setting $u(x) := F(\delta_x)$ for each $x \in X$. It is plain that u is continuous. One would then first verify the desired equation for simple lotteries (by induction), then use this fact to show that u is bounded, and finally establish the general case by using the fact that the set of all simple lotteries is dense in $\Delta(X)$.

A.2: Proofs of Main Results

Proof of Theorem 1

We only need to prove the "only if" part of the assertion. We begin with defining

$$\Lambda := \{(r, s) \in \Delta(X) \times \Delta(X) : \text{ not } r \mathbf{R} s\}$$

and

$$\mathcal{U}_{r,s} := \{u \in \mathbf{C}_{\mathrm{b}}(X) : \mathbf{E}(u, s) > \mathbf{E}(u, r)\}$$

for any $(r, s) \in \Lambda$. Finally, we put

$$\mathbb{U} := \{\mathcal{U}_{r,s} : (r, s) \in \Lambda\}.$$

Now, fix an arbitrary p and q in $\Delta(X)$. If $p\ \mathbf{R}\ q$ is false, then $(p,q) \in \Lambda$, so, trivially, $\mathsf{E}(u,q) > \mathsf{E}(u,p)$ for every $u \in \mathcal{U}_{p,q}$. Consequently,

$$p\ \mathbf{R}\ q \quad \text{if} \quad [\text{for every } \mathcal{U} \in \mathbb{U} \text{ there is a } u \in \mathcal{U} \text{ such that } \mathsf{E}(u,p) \geq \mathsf{E}(u,q)].$$

To prove the converse implication, assume that $p\ \mathbf{R}\ q$, and define $C := \text{cone}\{p-q\}$, which is a closed convex cone in $\text{ca}(X)$. Now take any $(r,s) \in \Lambda$. (We wish to find a $u \in \mathcal{U}_{r,s}$ such that $\mathsf{E}(u,p) \geq \mathsf{E}(u,q)$.) As $C \subseteq C(\mathbf{R})$ and $r\ \mathbf{R}\ s$ is false, part (c) of Lemma A.2 entails that $r - s \in \text{ca}(X)\backslash C$. We may then apply the Separating Hyperplane Theorem to find a continuous linear functional L on $\text{ca}(X)$ and a real number α such that $\inf L(C) \geq \alpha > L(r-s)$, that is, $\lambda(L(p) - L(q)) \geq \alpha > L(r) - L(s)$ for each $\lambda \geq 0$. Then, clearly, $L(p) \geq L(q)$. Moreover, $0 \geq \alpha$, and hence, $L(s) > L(r)$. Now apply the Affine Representation Lemma to conclude the proof. ∎

Proof of Theorem 2

Suppose that there is a coherent collection $\mathbb{U}$ of nonempty convex subsets of $\mathbf{C}_\text{b}(X)$ such that (3) holds for every p and q in $\Delta(X)$. It is obvious that $\mathbf{R}$ is an affine relation on $\Delta(X)$. Now take any p and q in $\Delta(X)$ such that $p\ \mathbf{R}\ q$ does not hold. By the representation of $\mathbf{R}$, this means that there is a $\mathcal{U}$ in $\mathbb{U}$ such that $\mathsf{E}(u,q) > \mathsf{E}(u,p)$ for each $u \in \mathcal{U}$. But, as $\mathbb{U}$ is coherent, $\mathcal{U} \cap \mathcal{V} \neq \emptyset$ for each $\mathcal{V} \in \mathbb{U}$. It follows that for every $\mathcal{V} \in \mathbb{U}$ there is a $v \in \mathcal{V}$ such that $\mathsf{E}(v,q) > \mathsf{E}(v,p)$, which means $q\ \mathbf{R}\ p$. In view of the arbitrariness of p and q, we conclude that $\mathbf{R}$ is complete.

To prove the "only if" part of the assertion, we continue using the notation introduced in the proof of Theorem 1. We wish to use the completeness of $\mathbf{R}$ to show that $\mathbb{U}$ is coherent. To this end, take any (r_1,s_1) and (r_2,s_2) in Λ. As $\mathbf{R}$ is complete, we have $s_1\ \mathbf{R}^>\ r_1$ and $s_2\ \mathbf{R}^>\ r_2$. As $\mathbb{U}$ is a coalitional minmax expected utility for $\mathbf{R}$, therefore, there must exist a $u \in \mathcal{U}_{r_1,s_1}$ with $\mathsf{E}(u,s_2) \geq \mathsf{E}(u,r_2)$, and a $v \in \mathcal{U}_{r_2,s_2}$ with $\mathsf{E}(v,s_1) \geq \mathsf{E}(v,r_1)$. Moreover, by definition of $\mathcal{U}_{r_1,s_1}$ and $\mathcal{U}_{r_2,s_2}$, we have $\mathsf{E}(u,s_1) > \mathsf{E}(u,r_1)$ and $\mathsf{E}(v,s_2) > \mathsf{E}(v,r_2)$, respectively, and it follows that $\frac{1}{2}u + \frac{1}{2}v \in \mathcal{U}_{r_1,s_1} \cap \mathcal{U}_{r_2,s_2}$. ∎

Proof of Theorem 3

To prove the "if" part of the theorem, assume that there is a collection $\mathbb{U}$ of nonempty compact subsets of $\mathbf{C}_\text{b}(X)$ such that (3) holds for every p and q in $\Delta(X)$. Clearly, this implies that $\mathbf{R}$ is an affine relation on $\Delta(X)$. To see that $\mathbf{R}$ is continuous as well, take any sequence (p_m,q_m) in $\mathbf{R}$ which converges to (p,q) (relative to the product topology on $\Delta(X) \times \Delta(X)$ induced by the topology of weak convergence). As $\Delta(X)$, and hence $\Delta(X) \times \Delta(X)$, is metrizable, we will be done if we can show that (p,q) belongs to $\mathbf{R}$. To this end, take an arbitrary $\mathcal{U} \in \mathbb{U}$. For each positive integer m, (3) implies that there is a $u_m \in \mathcal{U}$ such that $\mathsf{E}(u_m,p_m) \geq \mathsf{E}(u_m,q_m)$. Since $\mathcal{U}$ is compact, there is a subsequence, say, (u_{m_k}), of (u_m) which converges to some $u \in \mathcal{U}$. But it is well-known that E is a continuous real map on $\mathbf{C}(X) \times \Delta(X)$. It follows that $\mathsf{E}(u_m,p_m) \to \mathsf{E}(u,p)$ and $\mathsf{E}(u_m,q_m) \to \mathsf{E}(u,q)$, and hence, $\mathsf{E}(u,p) \geq \mathsf{E}(u,q)$. In view of the arbitrary choice of $\mathcal{U}$, and the representation (3), we conclude that $p\ \mathbf{R}\ q$, as we sought.

Conversely, assume that $\mathbf{R}$ is a continuous affine relation on $\Delta(X)$. If $\mathbf{R} = \Delta(X) \times \Delta(X)$, we are done by picking any constant real map u on X and setting $\mathbb{U} := \{\{u\}\}$, so we assume in what follows that $\mathbf{R}$ is a proper subset of $\Delta(X) \times \Delta(X)$. We may then define the binary relation $\mathbf{S}$ on $\Delta(X)$ by

$$p\ \mathbf{S}\ q \quad \text{iff} \quad \text{not } q\ \mathbf{R}\ p.$$

Clearly, $\mathbf{R}^> = \mathbf{S}^>$, while $\mathbf{S}$ is irreflexive and it satisfies the Independence Axiom. By part (a) of Lemma A.2, therefore, $p\ \mathbf{S}\ q$ iff $p - q \in C_0(\mathbf{S})$, for any $p,q \in \Delta(X)$. Moreover, $\mathbf{S}$ is an w^*-open subset of $\Delta(X) \times \Delta(X)$, for it is the complement of the w^*-closed set $\{(p,q) : q\ \mathbf{R}\ p\}$ in $\Delta(X) \times \Delta(X)$.

Claim 3.1. For any $(r,s) \in \mathbf{S}$, there is a subset C of $C_0(\mathbf{S})$ such that (i) $r - s \in C$; (ii) $C \cup \{\mathbf{0}\}$ is a convex cone in $\mathcal{S}(X)$; and (iii) C is bw^*-open in $\mathcal{S}(X)$.

Proof of Claim 3.1. Take any $r,s \in \Delta(X)$ with $r\ \mathbf{S}\ s$, and note that $r - s \in C_0(\mathbf{S})$. Since $\mathcal{S}(X)$ is a Hausdorff locally convex topological linear space relative to the bounded weak* topology, and we know from part (a) of Lemma A.3 that $C_0(\mathbf{S})$ is an open subset of this space, therefore, there is a convex bw^*-open set O in $\mathcal{S}(X)$ such that $r - s \in O \subseteq C_0(\mathbf{S})$. Put $C := \text{cone}(O)\backslash\{\mathbf{0}\}$, which obviously satisfies (i) and (ii) of the claim. As a vector μ in any convex cone in a topological linear space is in the interior of that cone iff $\lambda\mu$ is also in the interior of that cone for any $\lambda > 0$, it is evident that C satisfies (iii) as well. $\|$

For any $(r,s) \in \mathbf{S}$, let $C_{r,s}$ be a subset of $C_0(\mathbf{S})$ that satisfies the conditions (i), (ii) and (iii) of Claim 3.1. Notice that any μ in $C_0(\mathbf{S})$ can be written as $\lambda(r-s)$, and hence belongs to $C_{r,s}$, for some $\lambda > 0$ and $(r,s) \in \mathbf{S}$. Conversely, $r - s \in C_{r,s} \subseteq C_0(\mathbf{S})$ for any $(r,s) \in \mathbf{S}$. Therefore,

$$C_0(\mathbf{S}) = \bigcup\{C_{r,s} : (r,s) \in \mathbf{S}\}. \tag{10}$$

Next, take an arbitrary $(r,s) \in \mathbf{S}$, and define the binary relation $\succ_{r,s}$ on $\Delta(X)$ by

$$p \succ_{r,s} q \quad \text{iff} \quad p - q \in C_{r,s}. \tag{11}$$

By construction, $\succ_{r,s}$ is an asymmetric and transitive binary relation on $\Delta(X)$ which satisfies the Independence Axiom. Moreover, since $C_{r,s}$ is bw^*-open, $\succ_{r,s}$ is bw^*-open, and hence w^*-open, in $\Delta(X) \times \Delta(X)$. We may thus apply Theorem 1 of Evren (2014) to find a compact subset $\mathcal{U}_{r,s}$ of $\mathbf{C}(X)$ such that

$$p \succ_{r,s} q \quad \text{iff} \quad \mathsf{E}(u,p) > \mathsf{E}(u,q) \text{ for each } u \in \mathcal{U}_{r,s} \tag{12}$$

for any $p, q \in \Delta(X)$. We can in fact replace $\mathcal{U}_{r,s}$ with its closed and convex hull, which we denote by $\overline{\text{co}}(\mathcal{U}_{r,s})$, in this statement. Indeed, if p and q are two lotteries on X such that $p \succ_{r,s} q$, then compactness of $\mathcal{U}_{r,s}$ and (12) imply that

$$\varepsilon := \min_{u \in \mathcal{U}_{r,s}} (\mathsf{E}(u,p) - \mathsf{E}(u,q)) > 0,$$

and hence $\mathsf{E}(v,p) \geq \mathsf{E}(v,q) + \varepsilon$ for each $v \in \text{co}(\mathcal{U}_{r,s})$, which, in turn, entails that $\mathsf{E}(w,p) \geq \mathsf{E}(w,q) + \varepsilon$ for each $w \in \text{cl}(\text{co}(\mathcal{U}_{r,s}))$. Combining this observation with (12), therefore, we have

$$p \succ_{r,s} q \quad \text{iff} \quad \mathsf{E}(u,p) > \mathsf{E}(u,q) \text{ for each } u \in \overline{\text{co}}(\mathcal{U}_{r,s}) \tag{13}$$

for any $p, q \in \Delta(X)$.

Now for any lotteries p and q on X, by using (10), (11) and (13), we find that $q-p$ does not belong to $C_0(\mathbf{S})$ iff for every $(r,s) \in \mathbf{S}$ there is a $u \in \overline{\text{co}}(\mathcal{U}_{r,s})$ with $\mathsf{E}(u,p) \geq \mathsf{E}(u,q)$. But, by definition, $p \mathbf{R} q$ holds iff $q \mathbf{S} p$ does not, which happens exactly when $q - p$ does not belong to $C_0(\mathbf{S})$. Thus, setting $\mathbb{U}_0 := \{\overline{\text{co}}(\mathcal{U}_{r,s}) : (r,s) \in \mathbf{S}\}$, we may conclude:

$$p \mathbf{R} q \quad \text{iff} \quad [\text{for every } \mathcal{U} \in \mathbb{U}_0 \text{ there is a } u \in \mathcal{U} \text{ such that } \mathsf{E}(u,p) \geq \mathsf{E}(u,q)] \tag{14}$$

for any $p, q \in \Delta(X)$. Moreover, by Mazur's Compactness Theorem, $\overline{\text{co}}(\mathcal{U}_{r,s})$ is a compact set in $\mathbf{C}(X)$ for each $(r,s) \in \mathbf{S}$, so $\mathbb{U}_0$ lies within $\mathbf{k}(\mathbf{C}(X))$.[29]

To conclude our proof, we recall that compactness of X implies the separability of $\mathbf{C}(X)$. As the set of all nonempty compact subsets of a separable metric space is separable relative to the Hausdorff metric topology, it follows that $\mathbf{k}(\mathbf{C}(X))$, and hence $\mathbb{U}_0$, is separable. Pick any countable dense subset $\mathbb{U}_1$ of $\mathbb{U}_0$.

Claim 3.2. $\mathbb{U}_1$ is a coalitional minmax expected utility for $\mathbf{R}$.

Proof of Claim 3.2. Take any lotteries p and q on X. As $\mathbb{U}_1$ is a subset of $\mathbb{U}_0$, if $p \mathbf{R} q$, then we must have $\max_{v \in \mathcal{V}}(\mathsf{E}(v,p) - \mathsf{E}(v,q)) \geq 0$ for every $\mathcal{V} \in \mathbb{U}_1$. Suppose, then, $p \mathbf{R} q$ is false. Then, since $\mathbb{U}_0$ is a coalitional minmax expected utility for $\mathbf{R}$ (and every element of $\mathbb{U}_0$ is compact), there is a $\mathcal{U} \in \mathbb{U}_0$ such that

$$\varepsilon := \min_{u \in \mathcal{U}} (\mathsf{E}(u,q) - \mathsf{E}(u,p)) > 0.$$

Let us pick any $\mathcal{V}$ in $\mathbb{U}_1$ such that the Hausdorff distance between $\mathcal{U}$ and $\mathcal{V}$ is strictly less than $\frac{\varepsilon}{2}$. Then, for every $v \in \mathcal{V}$ there is a $u_v \in \mathcal{U}$ such that $\|v - u_v\| < \frac{\varepsilon}{2}$, which implies $|\mathsf{E}(v,r) - \mathsf{E}(u_v,r)| < \frac{\varepsilon}{2}$ for any $r \in \Delta(X)$. Therefore,

$$\begin{aligned} \mathsf{E}(v,q) - \mathsf{E}(v,p) &= (\mathsf{E}(v,q) - \mathsf{E}(u_v,q)) + (\mathsf{E}(u_v,q) - \mathsf{E}(u_v,p)) + (\mathsf{E}(u_v,p) - \mathsf{E}(v,p)) \\ &> -\tfrac{\varepsilon}{2} + \varepsilon - \tfrac{\varepsilon}{2} \end{aligned}$$

[29] Recall that $\mathbf{k}(\mathbf{C}(X))$ is the metric space of all nonempty compact subsets of $\mathbf{C}(X)$ relative to the Hausdorff metric induced by the sup-metric.

for any $v \in \mathcal{V}$. We have proved that if $p\ \mathbf{R}\ q$ is false, then there is a $\mathcal{V}$ in $\mathbb{U}_1$ such that $\mathsf{E}(v,q) > \mathsf{E}(v,p)$ for each $v \in \mathcal{V}$. This proves our claim. $\|$

Let us enumerate $\mathbb{U}_1$ as $\{\mathcal{V}_1, \mathcal{V}_2, ...\}$. In view of Claim 3.2,

$$p\ \mathbf{R}\ q \quad \text{iff} \quad \max_{v\in\mathcal{V}_i} (\mathsf{E}(v,p) - \mathsf{E}(v,q)) \geq 0 \text{ for each } i = 1, 2, ...$$

for every $p, q \in \Delta(X)$. On the other hand, for each positive integer i, $\mathcal{V}_i$ is a compact, and hence bounded, subset of $\mathbf{C}(X)$, and hence $K_i := \sup\{\|v\|_\infty : v \in \mathcal{V}_i\}$ is a real number. Furthermore, obviously,

$$p\ \mathbf{R}\ q \quad \text{iff} \quad \max_{v\in\mathcal{V}_i} \left(\frac{\mathsf{E}(v,p)}{iK_i+1} - \frac{\mathsf{E}(v,q)}{iK_i+1} \right) \geq 0 \text{ for each } i = 1, 2, ...$$

for every $p, q \in \Delta(X)$. Now let $\mathbf{0}_X$ denote the zero function on X, define $\mathbb{U} := \{\{\mathbf{0}_X\}, \frac{1}{K_1+1}\mathcal{V}_1, \frac{1}{2K_2+1}\mathcal{V}_2, , ...\}$, and note that

$$p\ \mathbf{R}\ q \quad \text{iff} \quad \max_{v\in\mathcal{U}} (\mathsf{E}(u,p) - \mathsf{E}(u,q)) \geq 0 \text{ for each } \mathcal{U} \in \mathbb{U}$$

for every $p, q \in \Delta(X)$. But it is readily checked that $\frac{1}{mK_m+1}\mathcal{V}_m \to^{\mathrm{H}} \{\mathbf{0}_X\}$, and hence $\mathbb{U}$ is a compact set in $\mathbf{k}(\mathbf{C}(X))$, and every member of it is convex. Proof of Theorem 3 is now complete. ■

Proof of Theorem 4

Let $\mathbb{U}$ be a compact and coherent collection of nonempty compact and convex subsets of $\mathbf{C}(X)$, which is a coalitional minmax expected utility for $\mathbf{R}$. Then, by Theorems 2 and 3, $\mathbf{R}$ is a continuous and complete affine relation on $\Delta(X)$. Conversely, suppose $\mathbf{R}$ is such a relation on $\Delta(X)$, and to focus on the nontrivial case, assume that $\mathbf{R}^> \neq \emptyset$. By Theorem 3, there is a compact collection $\mathbb{U}_0$ of nonempty compact and convex subsets of $\mathbf{C}(X)$ which is a coalitional minmax expected utility for $\mathbf{R}$. In fact, as the proof of Theorem 3 shows, we can choose this collection as countable. Let us now drop all members of this collection that contain a constant function, and denote the resulting collection by $\mathbb{U}_1$. As $\mathbf{R}^> \neq \emptyset$, $\mathbb{U}_1$ is not empty. Moreover, it is countable collection of nonempty compact and convex subsets of $\mathbf{C}(X)$ which is a coalitional minmax expected utility for $\mathbf{R}$.

For any nonempty subset $\mathcal{U}$ of $\mathbf{C}(X)$ we define

$$\Lambda(\mathcal{U}) := \bigcup_{(\alpha,\beta)\in\mathbb{R}_+\times\mathbb{R}} \alpha\mathcal{U} + \beta\{\mathbf{1}_X\} \quad \text{and} \quad \Lambda_0(\mathcal{U}) := \bigcup_{(\alpha,\beta)\in\mathbb{R}_{++}\times\mathbb{R}} \alpha\mathcal{U} + \beta\{\mathbf{1}_X\}.$$

(Notice that adjoining the constant maps on X to $\Lambda_0(\mathcal{U})$ we obtain precisely $\Lambda(\mathcal{U})$.) The following observation is the key step of the proof.

Claim 4.1. $\mathcal{U} \cap \Lambda(\mathcal{V}) \neq \emptyset$ for any $\mathcal{U}, \mathcal{V} \in \mathbb{U}_1$.

Proof of Claim 4.1. Take any $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{U}_1$, and suppose that $\mathcal{U} \cap \Lambda(\mathcal{V}) = \emptyset$. We may then strongly separate $\mathcal{U}$ and $\Lambda(\mathcal{V})$ – see Theorem 5.58 of Aliprantis and Border (2006) – to find a nonzero continuous linear functional L on $\mathbf{C}(X)$ such that $L(u) > 0 \geq L(v)$ for every $u \in \mathcal{U}$ and $v \in \Lambda(\mathcal{V})$. By the Riesz-Radon Representation Theorem, therefore, there is a nonzero $\mu \in \mathrm{ca}(X)$ such that

$$\int_X u d\mu > 0 \geq \int_X v d\mu \quad \text{for each } (u,v) \in \mathcal{U} \times \Lambda(\mathcal{V}). \tag{15}$$

On the other hand, $\mathcal{U} \cap \Lambda(\mathcal{V}) = \emptyset$ implies $\Lambda(\mathcal{U}) \cap \mathcal{V} = \emptyset$, so we may repeat this argument to find a nonzero $\nu \in \mathrm{ca}(X)$ such that

$$\int_X u d\nu \geq 0 > \int_X v d\nu \quad \text{for each } (u,v) \in \Lambda(\mathcal{U}) \times \mathcal{V}. \tag{16}$$

As 1_X and -1_X belong to both $\Lambda(\mathcal{U})$ and $\Lambda(\mathcal{V})$, (15) and (16) imply that $\mu(X) = 0 = \nu(X)$. We may then apply the Jordan Decomposition Theorem to the signed measure $\mu+\nu$ to find a constant $\lambda > 0$ and $p, q \in \Delta(X)$ such that $\mu + \nu = \lambda(p - q)$. In turn, combining (15) and (16) yields

$$\int_X u dp - \int_X u dq > 0 > \int_X v dp - \int_X v dq \quad \text{for each } (u,v) \in \mathcal{U} \times \mathcal{V}.$$

But, as $\mathbb{U}_1$ is a coalitional minmax expected utility for $\mathbf{R}$, this means that neither $p\ \mathbf{R}\ q$ nor $q\ \mathbf{R}\ p$ holds, contradicting $\mathbf{R}$ being complete. ‖

To focus on the less trivial case, we assume that $\mathbb{U}_1$ is not finite, and enumerate this collection as $\{\mathcal{U}_1, \mathcal{U}_2, ...\}$. For any integer $m > 1$, Claim 4.1 allows us to pick continuous real maps $u_{1,m}, ..., u_{m-1,m}$ on X such that

$$u_{1,m} \in \mathcal{U}_1 \cap \Lambda(\mathcal{U}_m),\ ...,\ u_{m-1,m} \in \mathcal{U}_{m-1} \cap \Lambda(\mathcal{U}_m).$$

We then define

$$\mathcal{V}_1 := \mathcal{U}_1,\ \mathcal{V}_2 := \mathrm{co}(\mathcal{U}_2 \cup \{u_{1,2}\}),\ \mathcal{V}_3 := \mathrm{co}(\mathcal{U}_3 \cup \{u_{1,3}, u_{2,3}\}),\ \mathcal{V}_4 := \mathrm{co}(\mathcal{U}_4 \cup \{u_{1,4}, u_{2,4}, u_{3,4}\}),\ ...$$

Recall $\bigcup \mathbb{U}_1$, hence no $\mathcal{U}_i$, may contain a constant map on X. Therefore, $\{u_{1,m}, ..., u_{i-1,i}\} \subseteq \Lambda_0(\mathcal{U}_i)$ for every positive integer i. As $\mathcal{U}_i \subseteq \Lambda_0(\mathcal{U}_i)$ and $\Lambda_0(\mathcal{U}_i)$ is convex, it follows that $\mathcal{V}_i \subseteq \Lambda_0(\mathcal{U}_i)$ for each $i \in \mathbb{N}$. In particular, no $\mathcal{V}_i$ may contain a constant map on X.

Now, by construction, $\mathbb{U}_2 := \{\mathcal{V}_1, \mathcal{V}_2, ...\}$ is a coherent collection of compact and convex subsets of $\mathbf{C}(X)$. It is also obvious that

$$p\ \mathbf{R}\ q \quad \text{implies} \quad \inf_{i \in \mathbb{N}} \max_{v \in \mathcal{V}_i} (\mathbf{E}(v, p) - \mathbf{E}(v, q)) \geq 0$$

for every p and q in $\Delta(X)$. Conversely, take any $p, q \in \Delta(X)$ such that for every positive integer i, there is a $v_i \in \mathcal{V}_i$ such that $\mathbf{E}(v_i, p) \geq \mathbf{E}(v_i, q)$. But for each $i \in \mathbb{N}$, given that $\mathcal{V}_i \subseteq \Lambda_0(\mathcal{U}_i)$, we have $v_i = \alpha_i u_i + \beta_i$ for some $u_i \in \mathcal{U}_i$ and $(\alpha_i, \beta_i) \in \mathbb{R}_{++} \times \mathbb{R}$, and hence $\mathbf{E}(u_i, p) \geq \mathbf{E}(u_i, q)$. As $\mathbb{U}_1$ is a coalitional minmax expected utility for $\mathbf{R}$, therefore, we find $p\ \mathbf{R}\ q$. Conclusion: $\mathbb{U}_2$ is a coherent and countable collection of nonempty compact and convex subsets of $\mathbf{C}(X)$ such that

$$p\ \mathbf{R}\ q \quad \text{iff} \quad \inf_{\mathcal{V} \in \mathbb{U}_2} \max_{v \in \mathcal{V}} (\mathbf{E}(v, p) - \mathbf{E}(v, q)) \geq 0 \tag{17}$$

for every p and q in $\Delta(X)$.

We now define $\mathbb{U} := \mathrm{cl}(\mathbb{U}_2)$, where the closure operator is applied within the metric space $\mathbf{k}(\mathbf{C}(X))$. As the Hausdorff limit of a sequence of compact and convex sets in a normed linear space must be convex, it is plain that every element of $\mathbb{U}$ is nonempty, compact and convex. Moreover, by construction, each $\mathcal{V} \in \mathbb{U}_2$ is a subset of $\overline{\mathrm{co}}(\bigcup \mathbb{U}_0)$. In other words, $\mathbb{U}_2$ is a subset of $\mathbf{k}(\overline{\mathrm{co}}(\bigcup \mathbb{U}_0))$. But $\bigcup \mathbb{U}_0$ is a compact subset of $\mathbf{C}(X)$, because $\mathbb{U}_0$ is a compact subset of $\mathbf{k}(\mathbf{C}(X))$. (See Theorem 2.5 of Michael (1951).) Therefore, by Mazur's Compactness Theorem, $\overline{\mathrm{co}}(\bigcup \mathbb{U}_0)$ is a compact subset of $\mathbf{C}(X)$. As the set of all nonempty compact subsets of a compact metric space is compact relative to the Hausdorff metric, therefore, $\mathbf{k}(\overline{\mathrm{co}}(\bigcup \mathbb{U}_0))$ is a compact subset of $\mathbf{k}(\mathbf{C}(X))$. Being a closed subset of $\mathbf{k}(\overline{\mathrm{co}}(\bigcup \mathbb{U}_0))$, therefore, $\mathbb{U}$ is compact in $\mathbf{k}(\mathbf{C}(X))$. In fact, this collection is also coherent. To see this, take any $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{U}$. Then, by definition of $\mathbb{U}$, there are sequences $(\mathcal{U}_m)$ and $(\mathcal{V}_m)$ in $\mathbb{U}_2$ such that $\mathcal{U}_m \to^{\mathrm{H}} \mathcal{U}$ and $\mathcal{V}_m \to^{\mathrm{H}} \mathcal{V}$. As $\mathbb{U}_2$ is coherent, there is a u_m in $\mathcal{U}_m \cap \mathcal{V}_m$ for each m. Since (u_m) is a sequence in the compact set $\overline{\mathrm{co}}(\bigcup \mathbb{U}_0)$, it has a subsequence, say, (u_{m_k}), that converges to some $u \in \mathbf{C}(X)$. As $\mathcal{U}_m \to^{\mathrm{H}} \mathcal{U}$ and $\mathcal{V}_m \to^{\mathrm{H}} \mathcal{V}$, it must be the case that u belongs to both $\mathcal{U}$ and $\mathcal{V}$. Thus, $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, and we conclude that $\mathbb{U}$ is coherent.

It remains to verify that $\mathbb{U}$ is a coalitional minmax expected utility for $\mathbf{R}$, that is,

$$p\ \mathbf{R}\ q \quad \text{if and only if} \quad \inf_{\mathcal{V} \in \mathbb{U}} \max_{v \in \mathcal{V}} (\mathbf{E}(v, p) - \mathbf{E}(v, q)) \geq 0 \tag{18}$$

for every p and q in $\Delta(X)$. Since $\mathbb{U}_2 \subseteq \mathbb{U}$, the "if" part of this statement is immediate from (17). Take, then, any two lotteries p and q on X with $p\ \mathbf{R}\ q$. Fix an arbitrary $\mathcal{U} \in \mathbb{U}$. By definition of $\mathbb{U}$, there is a sequence $(\mathcal{U}_m)$ in $\mathbb{U}_2$ such that $\mathcal{U}_m \to^{\mathrm{H}} \mathcal{U}$. As we apply the closure operator within $\mathbf{k}(\mathbf{C}(X))$, $\mathcal{U}$ is obviously a nonempty compact set in $\mathbf{C}(X)$. Finally, by (17), for each positive integer m, there is a $u_m \in \mathcal{U}_m$ such that $\mathbf{E}(u_m, p) \geq \mathbf{E}(u_m, q)$. Again, since (u_m) is a sequence in the compact set $\overline{\mathrm{co}}(\bigcup \mathbb{U}_0)$, it must have a subsequence, say, (u_{m_k}), that converges to some $u \in \mathbf{C}(X)$. Moreover, since $\mathcal{U}_{m_k} \to^{\mathrm{H}} \mathcal{U}$, it must be the case that $u \in \mathcal{U}$. As $\mathbf{E}(u_{m_k}, p) \geq \mathbf{E}(u_{m_k}, q)$ for each k, and $u_{m_k} \to u$ uniformly, therefore, we find $\mathbf{E}(u, p) \geq \mathbf{E}(u, q)$. We have proved (18). ■

Proof of Theorem 5

The "if" part of the assertion is readily proved, so we focus on its "only if" part. Assume that $\mathbf{R}$ is a continuous, strictly convex and affine preorder on $\Delta(X)$.

Claim 5.1. $\mathbf{R}$ is quasitransitive.

Proof of Claim 5.1. Take any $p, q, r \in \Delta(X)$ with $p\ \mathbf{R}^{>}\ q\ \mathbf{R}^{>}\ r$. Then, by the Independence Axiom,

$$\tfrac{1}{2}p + \tfrac{1}{2}r\ \mathbf{R}^{>}\ \tfrac{1}{2}q + \tfrac{1}{2}r \quad \text{and} \quad q\ \mathbf{R}^{>}\ \tfrac{1}{2}q + \tfrac{1}{2}r,$$

so, by strict convexity,

$$\tfrac{1}{2}q + \tfrac{1}{2}\left(\tfrac{1}{2}p + \tfrac{1}{2}r\right)\ \mathbf{R}^{>}\ \tfrac{1}{2}q + \tfrac{1}{2}r.$$

It then follows from the Independence Axiom that $\frac{1}{2}p + \frac{1}{2}r\ \mathbf{R}^{>}\ r$, and applying this axiom one more time yields $p\ \mathbf{R}^{>}\ r$, as we sought. $\|$

Our main assertion is trivially true if $\mathbf{R}^{>} = \emptyset$, so we assume that this is not the case. In view of Claim 5.1, then, $\mathbf{R}^{>}$ is an irreflexive and transitive binary relation on $\Delta(X)$. Moreover, as $\mathbf{R}$ is complete and continuous, $\mathbf{R}^{>}$ is open in $\Delta(X) \times \Delta(X)$. Therefore, we may apply Theorem 1 of Evren (2014) to find a nonempty $\mathcal{U} \subseteq \mathbf{k}(\mathbf{C}(X))$ such that

$$q\ \mathbf{R}^{>}\ p \quad \text{iff} \quad \mathbf{E}(u, q) > \mathbf{E}(u, p) \text{ for each } u \in \mathcal{U}$$

for every $p, q \in \Delta(X)$. Finally, we put $\mathcal{V} := \overline{\text{co}}(\mathcal{U})$, and note that this is a compact and convex subset of $\mathbf{C}(X)$, thanks to Mazur's Compactness Theorem. Moreover, for any $(q, p) \in \mathbf{R}^{>}$, compactness of $\mathcal{U}$ ensures that $\varepsilon := \min\{\mathbf{E}(u, q) - \mathbf{E}(u, p) : u \in \mathcal{U}\} > 0$ so that $\mathbf{E}(v, q) \geq \mathbf{E}(v, p) + \varepsilon > \mathbf{E}(v, p)$ for every $v \in \mathcal{V}$. Thus:

$$q\ \mathbf{R}^{>}\ p \quad \text{iff} \quad \mathbf{E}(v, q) > \mathbf{E}(v, p) \text{ for each } v \in \mathcal{V}$$

for every $p, q \in \Delta(X)$. As $\mathbf{R}$ is complete, this is equivalent to say that

$$p\ \mathbf{R}\ q \quad \text{iff} \quad \mathbf{E}(v, p) \geq \mathbf{E}(v, q) \text{ for some } v \in \mathcal{V}$$

for every $p, q \in \Delta(X)$. ■

Proof of Theorem 6

To see the "if" part, suppose that the said representation holds. Take any integer $k \geq 2$ and $p_1, ..., p_k \in \Delta(X)$ such that $p_1\ \mathbf{R}^{>} \cdots \mathbf{R}^{>}\ p_k$. Then, there must exist $\mathcal{U}_1, ..., \mathcal{U}_k$ in $\mathbb{U}$ such that for each $i \in [k-1]$ and $u \in \mathcal{U}_i$ we have $\mathbf{E}(u, p_i) > \mathbf{E}(u, p_{i+1})$. So, if $v \in \bigcap \mathbb{U}$, we have $\mathbf{E}(v, p_1) > \cdots > \mathbf{E}(v, p_k)$, and hence, $\mathbf{E}(v, p_1) > \mathbf{E}(v, p_k)$. It follows from the representation that $p_1\ \mathbf{R}\ p_k$. Conclusion: $\mathbf{R}$ is acyclic. In view of Theorem 4, we are done.

To prove the "only if" part of the theorem, we now assume that $\mathbf{R}$ is a continuous, complete and acyclic affine relation on $\Delta(X)$. We begin with a preliminary observation:

Claim 6.1. $\text{co}(C_0(\mathbf{R}^{>}))$ does not contain $\mathbf{0}$.

Proof of Claim 6.1. To derive a contradiction, let us assume that $\mathbf{0} \in \text{co}(C_0(\mathbf{R}^{>}))$. Then, there is an integer $k \geq 2$, nonnegative real numbers $\alpha_1, ..., \alpha_k$ and $\mu_1, ..., \mu_k \in C_0(\mathbf{R}^{>})$ such that $\sum^k \alpha_i = 1$ and $\sum^k \alpha_i \mu_i = \mathbf{0}$. In turn, by definition of $C_0(\mathbf{R}^{>})$, for each $i \in [k]$, there is a positive real number λ_i and $(p_i, q_i) \in \mathbf{R}^{>}$ such that $\mu_i = \lambda_i(p_i - q_i)$. Putting $\theta_j := \alpha_j \lambda_j / \sum^k \alpha_i \lambda_i$ for each j, therefore, we have

$$\sum_{i=1}^{k} \theta_i p_i = \sum_{i=1}^{k} \theta_i q_i.$$

We define

$$r_0 := \sum_{i=1}^{k} \theta_i p_i \quad \text{and} \quad r_j := \sum_{i=1}^{j} \theta_i q_i + \sum_{i=j+1}^{k} \theta_i p_i \text{ for each } j \in [k],$$

with the understanding that $\sum_{i=k+1}^{k} \theta_i p_i = \mathbf{0}$. Then, $r_1, ..., r_k \in \Delta(X)$, and by the Independence Axiom, $r_0\ \mathbf{R}^{>} \cdots \mathbf{R}^{>}\ r_k = r_0$, which contradicts $\mathbf{R}$ being acyclic. $\|$

Now, $\mathbf{R}^{>}$ is an irreflexive binary relation on $\Delta(X)$ that satisfies the Independence Axiom. Moreover, as $\mathbf{R}$ is complete, $\mathbf{R}^{>}$ is the complement of the closed set $\{(p,q) : q \mathbf{R} p\}$ in $\Delta(X) \times \Delta(X)$. It is thus an open subset of $\Delta(X) \times \Delta(X)$. Part (a) of Lemma A.3 thus applies to $\mathbf{R}^{>}$, that is, $C_0(\mathbf{R}^{>})$ is a bw^*-open subset of $\mathcal{S}(X)$. It follows that $\text{co}(C_0(\mathbf{R}^{>}))$ is a bw^*-open subset of $\mathcal{S}(X)$, because the convex hull of any open set in a topological linear space is open. In view of the Claim 6.1, and as any point outside an open convex set in a topological linear space can be strongly separated from that set – this is sometimes called Tukey's Separating Hyperplane Theorem – there is a continuous linear functional L on $\mathcal{S}(X)$ such that $L(\mu) > 0$ for each $\mu \in \text{co}(C_0(\mathbf{R}^{>}))$. By part (a) of Lemma A.2, therefore, $p \mathbf{R}^{>} q$ implies $L(p) > L(q)$, for every $p, q \in \Delta(X)$. We now apply the Hahn-Banach Extension Theorem (for locally convex topological linear spaces) to extend L to a continuous linear functional L^* on $\text{ca}(X)$. Then, $L^*|_{\Delta(X)}$ is a continuous and affine real map on $\Delta(X)$, so, by the Affine Representation Lemma, there is a $u \in \mathbf{C}(X)$ such that $L^*|_{\Delta(X)} = \mathsf{E}(u, \cdot)$, and hence

$$p \mathbf{R}^{>} q \quad \text{implies} \quad \mathsf{E}(u,p) > \mathsf{E}(u,q) \tag{19}$$

for every $p, q \in \Delta(X)$. Now let $\mathbb{V}$ be a closed subset of $\mathbf{C}(X)$ such that (14) holds for every $p, q \in \Delta(X)$, and define

$$\mathbb{U} := \{\text{co}(\mathcal{V} \cup \{u\}) : \mathcal{V} \in \mathbb{V}\}.$$

Obviously, $\bigcap \mathbb{U} \neq \emptyset$ and the "only if" part of (3) holds for every $p, q \in \Delta(X)$. Moreover, if $p \mathbf{R} q$ is false, then $q \mathbf{R}^{>} p$ (because $\mathbf{R}$ is complete), so there is a $\mathcal{V} \in \mathbb{V}$ such that $\mathsf{E}(v,q) > \mathsf{E}(v,p)$ for each $v \in \mathcal{V}$. Combining this with (19) shows that the "if" part of (3) holds for every $p, q \in \Delta(X)$ as well. ■

Proof of Proposition 5

The "if" part of the proposition is straightforward. To prove its "only if" part, let $\mathbf{R}$ be an FSD-transitive and continuous affine relation on $\Delta[0,1]$. (The case where $\mathbf{R}$ is SSD-transitive is analogously settled.) If $\mathbf{R} = \Delta[0,1] \times \Delta[0,1]$, we are done by picking any constant real map u on $[0,1]$ and setting $\mathbb{U} := \{\{u\}\}$, so we assume in what follows that $\mathbf{R}$ is a proper subset of $\Delta[0,1] \times \Delta[0,1]$.

For any $\mu \in C(\mathbf{R})$, we define

$$\Gamma(\mu) := \text{cl}(\text{cone}(\{\mu\} \cup C(\geq_{\text{FSD}}))),$$

where the closure operator is applied with respect to the bounded weak*-topology on $\mathcal{S}(X)$.

Claim 5.a. For every $\mu \in C(\mathbf{R})$, $\Gamma(\mu)$ is w^*-closed subset of $\mathcal{S}(X)$ such that $\Gamma(\mu) \subseteq C(\mathbf{R})$.

Proof of Claim 5.a. Take an arbitrary μ in $C(\mathbf{R})$, and notice that $\Gamma(\mu)$ is convex, because the closure of any convex set is convex in a Hausdorff topological linear space. But the Krein-Šmulian Theorem says that a convex subset of $\text{ca}(X)$ is w^*-closed iff it is bw^*-closed. As $\mathcal{S}(X)$ is closed in $\text{ca}(X)$ with respect to both w^*- and bounded w^*-topologies, therefore, a convex subset of $\mathcal{S}(X)$ is w^*-closed iff it is bw^*-closed in $\mathcal{S}(X)$. Conclusion: $\Gamma(\mu)$ is w^*-closed subset of $\mathcal{S}(X)$.

Now take any ν in $C(\geq_{\text{FSD}})$. Then, there are nonnegative numbers α and β such that $\mu = \alpha(p-q)$ and $\nu := \beta(r-s)$ for some lotteries p, q, r and s on $[0,1]$ such that $p \mathbf{R} q$ and $r \geq_{\text{FSD}} s$. Fix an arbitrary $\theta > 0$. Let us put $\lambda := \frac{\theta\alpha}{\theta\alpha+\beta}$, and note that by affinity of $\mathbf{R}$ and $\geq_{\text{FSD}}$, we have

$$\lambda p + (1-\lambda)r \mathbf{R} \lambda q + (1-\lambda)r \geq_{\text{FSD}} \lambda q + (1-\lambda)s.$$

So, by FSD-transitivity of $\mathbf{R}$, we find $\lambda p + (1-\lambda)r \mathbf{R} \lambda q + (1-\lambda)s$. By Lemma A.2.c, therefore,

$$\lambda(p-q) + (1-\lambda)(r-s) \in C(\mathbf{R}).$$

But

$$\theta\mu + \nu = (\theta\alpha+\beta)\left(\frac{\theta\alpha}{\theta\alpha+\beta}(p-q) + \frac{\beta}{\theta\alpha+\beta}(r-s)\right) = (\theta\alpha+\beta)\left(\lambda(p-q) + (1-\lambda)(r-s)\right),$$

and it follows that $\theta\mu+\nu \in C(\mathbf{R})$. In view of the arbitrariness of θ and ν, then, $\text{cone}(\{\mu\} \cup C(\geq_{\text{FSD}})) \subseteq C(\mathbf{R})$. Since, by part (b) of Lemma A.3, $C(\mathbf{R})$ is bw^*-closed in $\mathcal{S}(X)$, it follows that $\Gamma(\mu) \subseteq C(\mathbf{R})$. ‖

We now invoke Theorem 3 to find a compact collection $\mathbb{U}$ of nonempty compact and convex subsets of $\mathbf{C}[0,1]$, which is a coalitional minmax expected utility for $\mathbf{R}$.

Claim 5.b. For any $(p,q) \in \mathbf{R}$ and any $\mathcal{U} \in \mathbb{U}$, there is an increasing $u \in \mathcal{U}$ with $\mathsf{E}(u,p) \geq \mathsf{E}(u,q)$.

Proof of Claim 5.b. Take any $(p,q) \in \mathbf{R}$ and $\mathcal{U} \in \mathbb{U}$. Define the binary relation $\succsim$ on $\Delta[0,1]$ by

$$r \succsim s \quad \text{iff} \quad r - s \in \Gamma(p-q).$$

Since $\Gamma(p-q)$ is a w^*-closed convex cone in $\mathcal{S}(X)$, $\succsim$ is a continuous and affine preorder on $\Delta[0,1]$. (Note that $p \succsim q$.) We may thus apply the Dubra-Maccheroni-Ok Theorem to find a closed subset $\mathcal{V}$ of $\mathbf{C}[0,1]$ such that

$$r \succsim s \quad \text{iff} \quad \mathsf{E}(v,r) \geq \mathsf{E}(v,s) \text{ for each } v \in \mathcal{V}$$

for every r and s in $\Delta[0,1]$. Define

$$\mathcal{W} := \operatorname{cl}(\operatorname{cone}(\mathcal{V}) + \{\beta \mathbf{1}_X : \beta \in \mathbb{R}\}),$$

and note that $\mathcal{W}$ is a closed convex cone in $\mathbf{C}[0,1]$ such that

$$r \succsim s \quad \text{iff} \quad \mathsf{E}(w,r) \geq \mathsf{E}(w,s) \text{ for each } w \in \mathcal{W}$$

for every r and s in $\Delta[0,1]$. We wish to show that $\mathcal{W} \cap \mathcal{U}$ is nonempty. This is enough to complete the proof of our claim, because if there is a $u \in \mathcal{U}$ which belongs to $\mathcal{W}$, then we have $\mathsf{E}(u,p) \geq \mathsf{E}(u,q)$ (because $p \succsim q$ and $u \in \mathcal{W}$). Moreover, for any a and b in $[0,1]$ with $a \geq b$, we have $\delta_a \geq_{\text{FSD}} \delta_b$, and hence $\delta_a \succsim \delta_b$, which implies $u(a) \geq u(b)$ (because $u \in \mathcal{W}$), that is, u is increasing.

It remains to prove that $\mathcal{W} \cap \mathcal{U}$ is nonempty. The argument for this is analogous to the one we gave for Lemma 1. Assume $\mathcal{W} \cap \mathcal{U} = \emptyset$, and strongly separate $\mathcal{U}$ and $\mathcal{W}$ by a closed hyperplane in $\mathbf{C}[0,1]$. We use the classical Riesz-Radon Representation Theorem and the Jordan Decomposition Theorem, and the fact that $\mathbf{1}_{[0,1]}$ and $-\mathbf{1}_{[0,1]}$ belong to $\mathcal{W}$, to obtain two lotteries r and s on $[0,1]$ such that

$$\mathsf{E}(u,r) - \mathsf{E}(u,s) > 0 \geq \mathsf{E}(w,r) - \mathsf{E}(w,s) \quad \text{for every } (u,w) \in \mathcal{U} \times \mathcal{W}.$$

(The details are presented in the proof of Lemma 1.) As $\mathbb{U}$ is a coalitional minmax expected utility for $\mathbf{R}$, the first part of these inequalities implies that $s \; \mathbf{R} \; r$ does not hold. On the other hand, the second part of these inequalities entails $s \succsim r$. But then, $s - r \in \Gamma(p-q)$, so, by Claim 5.a, $s - r \in C(\mathbf{R})$, that is, $s \; \mathbf{R} \; r$, a contradiction. ‖

We now let $\mathcal{M}$ stand for the set of all continuous and increasing real maps on $[0,1]$, and put $\mathbb{V} := \{\mathcal{U} \cap \mathcal{M} : \mathcal{U} \in \mathbb{U}\}$. Notice that, in view of Claim 5.b,

$$\inf_{\mathcal{V} \in \mathbb{V}} \max_{v \in \mathcal{V}} (\mathsf{E}(v,p) - \mathsf{E}(v,q)) \geq 0 \quad \text{iff} \quad \inf_{\mathcal{U} \in \mathbb{U}} \max_{u \in \mathcal{U}} (\mathsf{E}(u,p) - \mathsf{E}(u,q)) \geq 0$$

for every p and q in $\Delta[0,1]$. Thus, $\mathbb{V}$ is a collection of nonempty compact and convex sets of continuous, increasing (and concave) real maps on $[0,1]$ which is a coalitional minmax expected utility for $\mathbf{R}$. Now put $\mathbb{W} := \operatorname{cl}(\mathbb{V})$, and note that $\mathbb{W}$ is also a collection of nonempty compact and convex sets of continuous, increasing (and concave) real maps on $[0,1]$. Moreover, compactness of $\mathbb{U}$ implies that of $\bigcup \mathbb{U}$, and hence that of $\mathbf{k}(\bigcup \mathbb{U})$, in $\mathbf{k}(\mathbf{C}[0,1])$. As $\mathbb{W}$ is a closed subset of $\mathbf{k}(\bigcup \mathbb{U})$, therefore, it is compact in $\mathbf{k}(\mathbf{C}[0,1])$. By routine arguments, it is also shown that $\mathbb{W}$ is a coalitional minmax expected utility for $\mathbf{R}$, and we are done. ■

APPENDIX B: Lexicographic Representation of Affine Preorders

As promised in Section 3.3.3, we now inquire how one may drop the hypothesis of completeness and continuity from the von Neumann-Morgenstern Theorem. To this end, we first show that it is possible to extend any given affine preorder on a lottery space to a complete affine preorder on that space. As the representation of the latter type of preorders is known, this will provide an answer to our present query.

Lemma B.1. *Let X be a metric space and $\succsim$ a binary relation on $\Delta(X)$. Then, $\succsim$ is an affine preorder on $\Delta(X)$ if, and only if, there is a collection $\mathcal{R}$ of complete affine preorders on $\Delta(X)$ such that $\succsim = \bigcap \mathcal{R}$.*

Proof. We only need to prove the "only if" part of the assertion. Let $\succsim$ be an affine preorder on $\Delta(X)$ and note that $C(\succsim)$ is a convex cone in $\mathcal{S}(X)$ by part (d) of Lemma A.2. If $\succsim$ equals $\Delta(X) \times \Delta(X)$, then the proof is completed upon setting $\mathcal{R} = \{\succsim\}$, so we assume this is not the case. This means that

$$\Lambda := \{(r, s) \in \Delta(X) \times \Delta(X) : \text{ not } r \succsim s\}$$

is not empty. Pick any $(r, s) \in \Lambda$, and note that part (c) of Lemma A.2 says that $r - s$ lies outside of $C(\succsim)$. By Lemma A.1, then, there is a convex cone C_{r-s} in $\text{ca}(X)$ such that (i) $C(\succsim) \subseteq C_{r-s}$; (ii) $r - s \in \text{ca}(X) \backslash C_{r-s}$; and (iii) $C_{r-s} \cup -C_{r-s} = \text{ca}(X)$. We define the binary relation $\trianglerighteq_{r,s}$ on $\Delta(X)$ by $p \trianglerighteq_{r,s} q$ iff $p - q \in C_{r-s}$. As C_{r-s} is a convex cone, $\trianglerighteq_{r,s}$ is an affine preorder on $\Delta(X)$. Moreover, by (iii), it is complete. Thus, $\mathcal{R} := \{\trianglerighteq_{r,s} : (r, s) \in \Lambda\}$ is a nonempty collection of complete affine preorders on $\Delta(X)$. Besides, given any p and q in $\Delta(X)$ with $p \succsim q$, we have $p - q \in C(\succsim) \subseteq C_{r-s}$, and hence $p \trianglerighteq_{r,s} q$, for each $(r, s) \in \Lambda$. Conversely, given any p and q in $\Delta(X)$ such that $p \succsim q$ is false, we have $(p, q) \in \Lambda$, so $p - q \in \text{ca}(X) \backslash C_{p-q}$, that is, $p \trianglerighteq_{p,q} q$ is false. Conclusion: $\succsim = \bigcap \mathcal{R}$. ■

Remark. This result is independently proved by Borie (2014), but the proof we provide for it here is substantially shorter. The main representation theorem of Shapley and Baucells (1998) is also related to this result.[30] In the setting of Lemma B.1, that result says that $\succsim$ is an affine preorder on $\Delta(X)$ such that the convex cone generated by $\{p - q : p \succsim q\}$ has a nonempty relative interior in the linear span of $\Delta(X) - \Delta(X)$ iff there is a collection $\mathcal{R}$ of complete affine preorders on $\Delta(X)$ such that (i) $\succsim = \bigcap \mathcal{R}$, and (ii) each $\trianglerighteq$ in $\mathcal{R}$ can be represented by an affine functional on $\Delta(X)$. □

While this is not apparent from its statement, Lemma B.1 actually allows us to look at an arbitrary affine preorder on $\Delta(X)$ also from the perspective of "expected multi-utility." Indeed, such a perspective is provided by Hausner and Wendel (1952) in the case of *complete* affine preorders.

The Hausner-Wendel Theorem is not widely known in decision theory, let alone microeconomics at large (despite the expository articles by Hausner (1954) and Fishburn (1971)). This is partly due to the fact that Hausner and Wendel (1952) is written in the language of ordered linear spaces; its terminology is outdated even in that context. As such, it is not immediately clear how to translate this result into the language of modern expected utility theory. In the hope of highlighting the depth of this result, and because we need it to give a "multi-utility" perspective to Lemma B.1, we provide one such translation next.

Let I be a loset, and recall that $\mathbb{R}^I$ stands for the set of all real functions on I. (This set is a (real) linear space relative to the usual (coordinatewise defined) addition and scalar multiplication operations.) For any $f \in \mathbb{R}^I$, the *support* of f is the set $\{i \in I : f(i) \neq 0\}$, which we denote by $\text{supp}(f)$. We define

$$\mathbb{R}^I_* := \{f \in \mathbb{R}^I : \text{supp}(f) \text{ is well-ordered}\},$$

and for any nonempty set A, say that an (affine) function $U : A \to \mathbb{R}^I$ is an **I-dimensional (affine) map** on A if $U(a) \in \mathbb{R}^I_*$ for each $a \in A$.[31] Finally, we define the binary relation $\geqslant_I$ on $\mathbb{R}^I_*$ as

$$f \geqslant_I g \quad \text{iff} \quad f = g \text{ or } f(j) > g(j),$$

where j is the minimum element of $\{i \in I : f(i) \neq g(i)\}$ with respect to the linear order of I. As the latter set is a subset of $\text{supp}(f) \cup \text{supp}(g)$, this relation is well-defined. Moreover, it is plain that $\geqslant_I$ is a partial order on $\mathbb{R}^I_*$. (Notice that when I is a set of the form $[n]$ with the usual ordering of integers, we may identify $\mathbb{R}^I_*$ with $\mathbb{R}^n$. Thus, in that case, the notion of an I-dimensional map on A reduces simply to that of $\mathbb{R}^n$-valued map on A, and $\geqslant_I$ to the familiar lexicographic order on $\mathbb{R}^n$.)

With these preliminaries at hand, we may state the embedding theorem of Hausner and Wendel (1952) in the context of expected utility theory as follows:

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[30] A revised version of this working paper is published as Baucells and Shapley (2008), but that version focuses much more on "group preferences," and it does not report this representation theorem.

[31] If A is a loset (with the underlying linear order $\geqslant$), and B is a nonempty subset of A such that every nonempty subset of B possesses a minimum element with respect to $\geqslant$, we say that B is **well ordered** (by $\geqslant$).

The Hausner-Wendel Theorem. *Let X be a metric space and $\succsim$ a binary relation on $\Delta(X)$. Then, $\succsim$ is a complete and affine preorder on $\Delta(X)$ if, and only if, there exist a loset I and an I-dimensional affine function U on $\Delta(X)$ such that*

$$p \succsim q \quad \text{iff} \quad U(p) \geqslant_I U(q)$$

for every p and q in $\Delta(X)$.[32]

As the representing function U found in this theorem is vector-valued, we can interpret the component functions of U as "utility functions." These "utility functions" fit well to the expected utility paradigm, as each of them is affine. (These functions are not guaranteed to possess an "expected utility" form, but this is a fair price to pay for not imposing any continuity conditions on $\succsim$.) Consequently, we may think of an agent whose preference relation over lotteries on X is a complete and affine preorder as one who again has "multiple selves," each of whom evaluates lotteries by means of an affine utility function. In this case the agent's preference relation arises from lexicographically aggregating these utility functions. As such, the Hausner-Wendel Theorem is very much a "multi-utility theorem," even though it considers complete preference relations.

While this matter is not at all discussed in Hausner-Wendel (1952) and the literature that has followed it, we can easily see how the theorem above would modify if we were to drop completeness as a hypothesis. Indeed, combining this result with Lemma B.1 readily yields the following "multi-utility" theorem.

Theorem B.1. *Let X be a metric space and $\succsim$ a binary relation on $\Delta(X)$. Then, $\succsim$ is an affine preorder on $\Delta(X)$ if, and only if, there exist a nonempty collection $\mathcal{I}$ of losets and a map $U : \Delta(X) \times \mathcal{I} \to \bigcup_{I \in \mathcal{I}} \mathbb{R}^I$ such that $U(\cdot, I)$ is an I-dimensional affine function on $\Delta(X)$ for each $I \in \mathcal{I}$, and*

$$p \succsim q \quad \text{iff} \quad U(p, I) \geqslant_I U(q, I) \text{ for each } I \in \mathcal{I}$$

for every p and q in $\Delta(X)$.

While it is technically a bit more complicated than the one we obtained in Theorem 1, this representation notion too enjoys a similar interpretation. It suggests viewing a person who has a reflexive and transitive preference relation on $\Delta(X)$ which satisfies the Independence Axiom as arising from the aggregation of coalitions of "selves." In this case, the preference relation of these "selves" need not be of an expected utility form (due to lack of continuity), but they are represented by an affine utility function over lotteries (as in Shapley and Baucells (1998)). Within each coalition, these utility functions are aggregated lexicographically (with respect to endogenously found linear orders, which may differ from coalition to coalition). So, "coalitional preferences" in this *as if* interpretation are complete. In turn, the agent prefers one lottery p over another lottery q iff every coalition of her "selves" say that p is better than q. In the case of disagreement between the coalitions, the agent remains indecisive.

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[32] Hausner and Wendel (1952) assume the antisymmetry of $\succsim$ in their paper, but the statement we give here is easily deduced from their main theorem by the method of passing to the quotient.

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